UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MSG 388 – Mathematical Algorithms for Computer Graphics
[Algoritma Matematik untuk Grafik Komputer]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Derive the Bézier form of a polynomial function $y(x) = (1-x)(1+2x^2)$, where $x \in [0,\ 1]$.

(b) Let $\boldsymbol{R}(u)$ be a parametric spline curve defined as

$$\boldsymbol{R}(u) = \begin{cases} \boldsymbol{P}(u), & 0 \le u \le 1 \\ \boldsymbol{Q}(u), & 1 \le u \le 3, \end{cases}$$

where $\boldsymbol{P}(u)$ and $\boldsymbol{Q}(u)$ can be represented locally as

$$\boldsymbol{P}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}(1-t)^2 + \begin{pmatrix} a \\ b \end{pmatrix}(1-t)t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}t^2,$$

$$\boldsymbol{Q}(t) = \begin{pmatrix} c \\ d \end{pmatrix}(1-t)^3 + \begin{pmatrix} 7 \\ 3 \end{pmatrix}(1-t)^2 t + \begin{pmatrix} e \\ f \end{pmatrix}(1-t)t^2 + \begin{pmatrix} 3 \\ 0 \end{pmatrix}t^3,$$

with parameter $t \in [0,\ 1]$ and a, b, c, d, e, $f \in \mathbb{R}$. If $\boldsymbol{R}(u)$ is curvature continuous (G^2) at $u = 1$ with

$$\frac{\mathrm{d}^2\boldsymbol{Q}}{\mathrm{d}u^2}(1) = \frac{\mathrm{d}^2\boldsymbol{P}}{\mathrm{d}u^2}(1) + 2\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1),$$

evaluate the points $(a,\ b)$, $(c,\ d)$ and $(e,\ f)$.

(c) A biquadratic Bézier surface is defined by

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \boldsymbol{C}_{i,\,j}\, B_j^2(v)\, B_i^2(u), \qquad 0 \le u \le 1,\ \ 0 \le v \le 1,$$

where B_s^2, $s = 0,\ 1,\ 2$, are the Bernstein polynomials of degree 2,

$$B_s^2(t) = \frac{2}{s!(2-s)!}(1-t)^{2-s}t^s, \qquad 0 \le t \le 1.$$

$\boldsymbol{C}_{i,\,j}$ are the Bézier points given as

$\boldsymbol{C}_{0,0} = (0,\ 0,\ 1)$, $\boldsymbol{C}_{1,0} = (1,\ 0,\ 3)$, $\boldsymbol{C}_{2,0} = (2,\ 0,\ 1)$,

$\boldsymbol{C}_{0,1} = (0,\ 1,\ 2)$, $\boldsymbol{C}_{1,1} = (1,\ 1,\ 1)$, $\boldsymbol{C}_{2,1} = (2,\ 1,\ 2)$,

$\boldsymbol{C}_{0,2} = (0,\ 2,\ 1)$, $\boldsymbol{C}_{1,2} = (1,\ 2,\ 3)$, $\boldsymbol{C}_{2,2} = (2,\ 2,\ 1)$.

Use the de Casteljau algorithm to evaluate the derivative $\dfrac{\partial^2 \boldsymbol{S}}{\partial u\, \partial v}(u,\ v)$ at $(u, v) = (0.5,\ 0.25)$.

[100 marks]

1. (a) *Terbitkan perwakilan Bézier bagi fungsi polinomial* $y(x) = (1-x)(1+2x^2)$, *dengan* $x \in [0, 1]$.

(b) *Katakan* $\boldsymbol{R}(u)$ *ialah suatu lengkung splin berparameter yang ditakrif sebagai*

$$\boldsymbol{R}(u) = \begin{cases} \boldsymbol{P}(u), & 0 \le u \le 1 \\ \boldsymbol{Q}(u), & 1 \le u \le 3, \end{cases}$$

di mana $\boldsymbol{P}(u)$ *dan* $\boldsymbol{Q}(u)$ *dapat diwakili secara setempat seperti*

$$\boldsymbol{P}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}(1-t)^2 + \begin{pmatrix} a \\ b \end{pmatrix}(1-t)\,t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2,$$

$$\boldsymbol{Q}(t) = \begin{pmatrix} c \\ d \end{pmatrix}(1-t)^3 + \begin{pmatrix} 7 \\ 3 \end{pmatrix}(1-t)^2\, t + \begin{pmatrix} e \\ f \end{pmatrix}(1-t)\,t^2 + \begin{pmatrix} 3 \\ 0 \end{pmatrix} t^3,$$

dengan parameter $t \in [0, 1]$ *dan* a, b, c, d, e, $f \in \mathbb{R}$. *Jika* $\boldsymbol{R}(u)$ *berkeselanjaran kelengkungan (*G^2*) pada* $u = 1$ *dengan*

$$\frac{\mathrm{d}^2\boldsymbol{Q}}{\mathrm{d}u^2}(1) = \frac{\mathrm{d}^2\boldsymbol{P}}{\mathrm{d}u^2}(1) + 2\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1),$$

nilaikan titik-titik (a, b), (c, d) *dan* (e, f).

(c) *Suatu permukaan Bézier bikuadratik ditakrif oleh*

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \boldsymbol{C}_{i,j}\, B_j^2(v)\, B_i^2(u), \qquad 0 \le u \le 1,\ 0 \le v \le 1,$$

dengan B_s^2, $s = 0, 1, 2$, *polinomial Bernstein berdarjah* 2,

$$B_s^2(t) = \frac{2}{s!(2-s)!}(1-t)^{2-s} t^s, \qquad 0 \le t \le 1.$$

$\boldsymbol{C}_{i,j}$ *adalah titik Bézier yang diberikan sebagai*

$\boldsymbol{C}_{0,0} = (0, 0, 1)$, $\boldsymbol{C}_{1,0} = (1, 0, 3)$, $\boldsymbol{C}_{2,0} = (2, 0, 1)$,

$\boldsymbol{C}_{0,1} = (0, 1, 2)$, $\boldsymbol{C}_{1,1} = (1, 1, 1)$, $\boldsymbol{C}_{2,1} = (2, 1, 2)$,

$\boldsymbol{C}_{0,2} = (0, 2, 1)$, $\boldsymbol{C}_{1,2} = (1, 2, 3)$, $\boldsymbol{C}_{2,2} = (2, 2, 1)$.

Gunakan algoritma de Casteljau untuk menilai terbitan $\dfrac{\partial^2 \boldsymbol{S}}{\partial u\, \partial v}(u, v)$ *pada* $(u, v) = (0.5, 0.25)$.

[100 markah]

2. (a) Let $\boldsymbol{P}(u)=\sum_{i=0}^{n}\boldsymbol{D}_i N_i^k(u)$ be a k-order B-spline curve defined by basis functions

$$N_i^k(u)=\frac{u-u_i}{u_{i+k-1}-u_i}N_i^{k-1}(u)+\frac{u_{i+k}-u}{u_{i+k}-u_{i+1}}N_{i+1}^{k-1}(u), \qquad k>1,$$

and

$$N_i^1(u)=\begin{cases}1, & u_i\le u<u_{i+1}\\ 0, & \text{otherwise,}\end{cases}$$

over a knot vector $\boldsymbol{u}=(u_0, u_1, \ldots, u_{n+k})$ with the de Boor points $\boldsymbol{D}_i\in\mathbb{R}^2$, $i=0$, $1, \ldots, n$.

(I) Suppose $k=4$, $\boldsymbol{u}=(0, 1, 2, 3, 4, 5, 6, 7)$, $\boldsymbol{D}_0=(1, 1)$, $\boldsymbol{D}_1=(1, 2)$, $\boldsymbol{D}_2=(2, 2)$ and $\boldsymbol{D}_3=(2, 1)$. Evaluate the second derivative of $\boldsymbol{P}(u)$ at $u=3.5$.

(II) Suppose $k=4$ and $\boldsymbol{u}=(-3, -2, -1, 0, 1, 2, 3, 4)$. Determine the de Boor points $\boldsymbol{D}_i$, $i=0, 1, \ldots, 3$, such that the curve

$$\boldsymbol{P}(u)=\begin{pmatrix}1\\2\end{pmatrix}N_1^3(u)+\begin{pmatrix}2\\1\end{pmatrix}N_2^3(u)+\begin{pmatrix}3\\2\end{pmatrix}N_3^3(u), \qquad u\in[0, 1].$$

(b) A biquadratic B-spline surface

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{D}_{i,j}N_j^3(v)N_i^3(u), \qquad 0\le u\le 1, \quad 0\le v\le 1,$$

is defined over the knot vectors $\boldsymbol{u}=(-1.5, -0.5, 0, 1, 1.75, 2.5)$ and $\boldsymbol{v}=(-2, -1, 0, 1, 2, 3)$ with the de Boor points

$\boldsymbol{D}_{0,0}=(0, 0, 1)$, $\boldsymbol{D}_{1,0}=(1, 0, 3)$, $\boldsymbol{D}_{2,0}=(2, 0, 1)$,
$\boldsymbol{D}_{0,1}=(0, 1, 2)$, $\boldsymbol{D}_{1,1}=(1, 1, 1)$, $\boldsymbol{D}_{2,1}=(2, 1, 2)$,
$\boldsymbol{D}_{0,2}=(0, 2, 1)$, $\boldsymbol{D}_{1,2}=(1, 2, 3)$, $\boldsymbol{D}_{2,2}=(2, 2, 1)$.

Use the de Boor algorithm to evaluate $\boldsymbol{S}(u,v)$ at $(u,v)=(0.5, 0.5)$.

[100 marks]

2. (a) *Katakan* $\boldsymbol{P}(u) = \sum_{i=0}^{n} \boldsymbol{D}_i N_i^k(u)$ *ialah suatu lengkung splin-B berperingkat* k *yang ditakrif oleh fungsi-fungsi asas*

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad k > 1,$$

dan

$$N_i^1(u) = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \textit{di tempat lain}, \end{cases}$$

ke atas suatu vektor simpulan $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n+k})$ *dengan titik-titik de Boor* $\boldsymbol{D}_i \in \mathbb{R}^2$, $i = 0, 1, \ldots, n$.

(I) *Andaikan* $k = 4$, $\boldsymbol{u} = (0, 1, 2, 3, 4, 5, 6, 7)$, $\boldsymbol{D}_0 = (1, 1)$, $\boldsymbol{D}_1 = (1, 2)$, $\boldsymbol{D}_2 = (2, 2)$ *dan* $\boldsymbol{D}_3 = (2, 1)$. *Nilaikan terbitan kedua bagi* $\boldsymbol{P}(u)$ *pada* $u = 3.5$.

(II) *Andaikan* $k = 4$ *dan* $\boldsymbol{u} = (-3, -2, -1, 0, 1, 2, 3, 4)$. *Tentukan titik-titik de Boor* $\boldsymbol{D}_i$, $i = 0, 1, \ldots, 3$, *supaya lengkung*

$$\boldsymbol{P}(u) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} N_1^3(u) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} N_2^3(u) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} N_3^3(u), \qquad u \in [0, 1].$$

(b) *Suatu permukaan splin-B bikuadratik*

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \boldsymbol{D}_{i,j} N_j^3(v) N_i^3(u), \qquad 0 \le u \le 1, \quad 0 \le v \le 1,$$

ditakrif ke atas vektor-vektor simpulan $\boldsymbol{u} = (-1.5, -0.5, 0, 1, 1.75, 2.5)$ *dan* $\boldsymbol{v} = (-2, -1, 0, 1, 2, 3)$ *dengan titik-titik de Boor*

$$\boldsymbol{D}_{0,0} = (0, 0, 1), \quad \boldsymbol{D}_{1,0} = (1, 0, 3), \quad \boldsymbol{D}_{2,0} = (2, 0, 1),$$

$$\boldsymbol{D}_{0,1} = (0, 1, 2), \quad \boldsymbol{D}_{1,1} = (1, 1, 1), \quad \boldsymbol{D}_{2,1} = (2, 1, 2),$$

$$\boldsymbol{D}_{0,2} = (0, 2, 1), \quad \boldsymbol{D}_{1,2} = (1, 2, 3), \quad \boldsymbol{D}_{2,2} = (2, 2, 1).$$

Gunakan algoritma de Boor untuk menilai $\boldsymbol{S}(u, v)$ *pada* $(u, v) = (0.5, 0.5)$.

[100 markah]

3. (a) Given a rational quadratic Bézier curve

$$\boldsymbol{R}(t) = \frac{\boldsymbol{C}_0 B_0^2(t) + w\boldsymbol{C}_1 B_1^2(t) + \boldsymbol{C}_2 B_2^2(t)}{B_0^2(t) + wB_1^2(t) + B_2^2(t)}, \qquad t \in [0, 1]$$

where $B_i^2(t)$ are the Bernstein polynomials of degree 2, the coefficients $\boldsymbol{C}_i \in \mathbb{R}^2$, $i = 0, 1, 2$, are the Bézier points and $w \geq 0$.

(I) Suppose $\boldsymbol{C}_0$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ are non-collinear points, state the conditions on w such that $\boldsymbol{R}(t)$ is

(i) an elliptic arc;

(ii) a parabolic arc;

(iii) a hyperbolic arc.

(II) If $\boldsymbol{R}(t)$ represents a circular arc $x^2 + y^2 - 4 = 0$, where $x \in [1, 2]$, determine the coefficients $\boldsymbol{C}_i$, $i = 0, 1, 2$, and w.

(b) A Bézier triangle of degree n is defined by

$$\boldsymbol{S}(u, v, w) = \sum_{\substack{i, j, k \geq 0 \\ i+j+k=n}} \boldsymbol{C}_{i,j,k} B_{i,j,k}^n(u, v, w), \quad 0 \leq u, v, w \leq 1, \quad u + v + w = 1$$

where $B_{i,j,k}^n(u, v, w)$ are the generalised Bernstein polynomials

$$B_{i,j,k}^n(u, v, w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k,$$

and $\boldsymbol{C}_{i,j,k} \in \mathbb{R}^3$ are the Bézier points. Let $n = 2$ and the Bézier points are given as

$\boldsymbol{C}_{2,0,0} = (1, 1, 1)$,

$\boldsymbol{C}_{1,1,0} = (1.5, 2, 1)$, $\boldsymbol{C}_{1,0,1} = (2, 1.5, 1)$,

$\boldsymbol{C}_{0,2,0} = (2, 3, 2)$, $\boldsymbol{C}_{0,1,1} = (2.5, 2.5, 2)$, $\boldsymbol{C}_{0,0,2} = (3, 2, 3)$.

(I) If $\boldsymbol{P} = (2, 2, z)$ is a point on the surface $\boldsymbol{S}(u, v, w)$, use the de Casteljau algorithm to evaluate z.

(II) Let $\boldsymbol{d} = (0.5, 0.5, -1)$ be a vector written in barycentric coordinates. Evaluate the directional derivative of $\boldsymbol{S}(u, v, w)$ with respect to vector $\boldsymbol{d}$ at point $\boldsymbol{C}_{2,0,0}$.

[100 marks]

3. (a) *Diberi lengkung Bézier kuadratik nisbah*

$$\boldsymbol{R}(t) = \frac{\boldsymbol{C}_0 B_0^2(t) + w\boldsymbol{C}_1 B_1^2(t) + \boldsymbol{C}_2 B_2^2(t)}{B_0^2(t) + wB_1^2(t) + B_2^2(t)}, \qquad t \in [0, 1]$$

di mana $B_i^2(t)$ *ialah polinomial Bernstein berdarjah* 2*, koefisien* $\boldsymbol{C}_i \in \mathbb{R}^2$, $i = 0$, $1, 2$*, adalah tititk-titik Bézier dan* $w \geq 0$.

(I) *Andaikan* $\boldsymbol{C}_0$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ *adalah titik-titik tak kolinear, nyatakan syarat-syarat pada* w *supaya* $\boldsymbol{R}(t)$ *ialah*

(i) *lengkok eliptik;*

(ii) *lengkok parabolik;*

(iii) *lengkok hiperbolik.*

(II) *Jika* $\boldsymbol{R}(t)$ *mewakili lengkok bulatan* $x^2 + y^2 - 4 = 0$, *dengan* $x \in [1, 2]$, *tentukan koefisien-koefisien* $\boldsymbol{C}_i$, $i = 0, 1, 2$, *dan* w.

(b) *Suatu segi tiga Bézier berdarjah* n *ditakrif oleh*

$$\boldsymbol{S}(u, v, w) = \sum_{\substack{i, j, k \geq 0 \\ i+j+k=n}} \boldsymbol{C}_{i,j,k} B_{i,j,k}^n(u, v, w), \qquad 0 \leq u, v, w \leq 1, \quad u + v + w = 1$$

di mana $B_{i,j,k}^n(u, v, w)$ *ialah polinomial Bernstein teritlak*

$$B_{i,j,k}^n(u, v, w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k,$$

dan $\boldsymbol{C}_{i,j,k} \in \mathbb{R}^3$ *adalah titik-titik Bézier. Katakan* $n = 2$ *dan titik–titik Bézier diberikan sebagai*

$\boldsymbol{C}_{2,0,0} = (1, 1, 1)$,

$\boldsymbol{C}_{1,1,0} = (1.5, 2, 1)$, $\quad \boldsymbol{C}_{1,0,1} = (2, 1.5, 1)$,

$\boldsymbol{C}_{0,2,0} = (2, 3, 2)$, $\quad \boldsymbol{C}_{0,1,1} = (2.5, 2.5, 2)$, $\quad \boldsymbol{C}_{0,0,2} = (3, 2, 3)$.

(I) *Jika* $\boldsymbol{P} = (2, 2, z)$ *ialah suatu titik pada permukaan* $\boldsymbol{S}(u, v, w)$, *gunakan algoritma de Casteljau untuk menilai* z.

(II) *Andaikan* $\boldsymbol{d} = (0.5, 0.5, -1)$ *ialah suatu vektor yang ditulis dalam koordinat baripusat. Nilaikan terbitan berarah bagi* $\boldsymbol{S}(u, v, w)$ *terhadap vektor* $\boldsymbol{d}$ *pada titik* $\boldsymbol{C}_{2,0,0}$.

[100 markah]

- ooo O ooo -