UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) The two events A and B cannot be simultaneously mutually exclusive and independent. Assume that $P(A) > 0$ and $P(B) > 0$. Show that A and B cannot be independent if they are mutually exclusive.

(b) X is a random variable with p.d.f $f(x) = \frac{1}{2}x^2 e^{-x}$, $0 < x < \infty$

(i) Find the moment generating function $M(t)$.

(ii) Find the values of μ and σ^2

(c) Let the p.d.f of Y be given as $f(y) = \frac{c}{y^2}$, $1 < y < \infty$

(i) Calculate the value of c so that $f(y)$ is a p.d.f.

(ii) Show that $E(Y)$ is not finite. (i.e $E(Y)$ does not exist).

[100 marks]

2. (a) X and Y are two random variables. The joint p.d.f. of X and Y is given by $f(x, y) = \frac{1}{8}$, $0 \le y \le 4$, $y \le x \le y + 2$

(i) Find the marginal p.d.f of X, $f_1(x)$ and marginal p.d.f of Y, $f_2(y)$

(ii) Find the conditional p.d.f Y, given that $X = x$, i.e. $h(y|x)$

(iii) Compute $E(Y|x)$

(b) X and Y are two independent random variables with normal distribution, i.e. $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \sigma^2)$ respectively. U and V are defined to be $U = X + Y$ and $V = X - Y$ respectively.

(i) Find $f_{X,Y}(x, y)$

(ii) Find $f_{U,V}(u, v)$

[Hint : Apply the Jacobian Factorization Theorem]

[100 marks]

1. (a) *Dua peristiwa A dan B tidak boleh saling ekslusif dan tak bersandar secara serentak. Andaikan* $P(A) > 0$ *dan* $P(B) > 0$. *Tunjukkan bahawa A dan B tidak boleh tak bersandar jika kedua-dua peristiwa itu saling ekslusif.*

   (b) X *ialah pembolehubah rawak dengan fungsi ketumpatan kebarangkalian (f.k.k)* $f(x) = \frac{1}{2}x^2e^{-x}$, $0 < x < \infty$

   (i) *Cari fungsi penjanaan momen,* $M(t)$.

   (ii) *Cari nilai-nilai untuk* μ *dan* σ^2

   (c) *Biarkan f.k.k untuk Y diberi sebagai* $f(y) = \frac{c}{y^2}$, $1 < y < \infty$

   (i) *Kira nilai c supaya* $f(y)$ *ialah f.k.k.*

   (ii) *Tunjukkan bahawa* $E(Y)$ *tidak terhingga (iaitu* $E(Y)$ *tidak wujud).*

[100 markah]

2. (a) *X dan Y ialah dua pembolehubah rawak. f.k.k tercantum untuk X dan Y diberi sebagai* $f(x, y) = \frac{1}{8}$, $0 \le y \le 4$, $y \le x \le y + 2$.

   (i) *Cari f.k.k sut untuk X,* $f_1(x)$ *dan f.k.k sut untuk Y,* $f_2(y)$

   (ii) *Cari f.k.k bersyarat untuk Y, diberi* $X = x$, *iaitu* $h(y|x)$

   (iii) *Kirakan* $E(Y|x)$

   (b) *X dan Y merupakan dua pembolehubah rawak tak bersandar dengan taburan normal iaitu* $X \sim N(\mu, \sigma^2)$ *dan* $Y \sim N(\gamma, \sigma^2)$ *masing-masing. U dan V didefinisikan sebagai* $U = X + Y$ *dan* $V = X - Y$ *masing-masing.*

   (i) *Cari* $f_{X,Y}(x, y)$

   (ii) *Cari* $f_{U,V}(u, v)$

   [Petunjuk : gunakan Teorem Pemfaktoran Jakobian]

[100 markah]

3. (a) Let $X_1,\ldots,X_n$ be a random sample from a population with mean μ and variable σ^2. Show that

$$E\left[\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{\sigma}\right]=0 \quad \text{and} \quad \operatorname{var}\left[\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{\sigma}\right]=1$$

so that the normalization of $\overline{X}_n$ has the limiting $N(0,1)$ distribution.

(b) Show that if $\sqrt{n}\left(Y_n-\mu\right)\rightarrow N\left(0,\sigma^2\right)$ in distribution, then $Y_n\rightarrow\mu$ in probability.

(c) Let $X_1,\ldots,X_n$ be a random sample from a population with p.d.f

$$f\left(x|\theta\right)=\theta x^{\theta-1}, \quad 0<x<1, \quad \theta>0$$

(i) Is $\sum X_i$ sufficient for θ?

(ii) Find a complete sufficient statistic for θ.

[100 marks]

4. (a) Let $X_1,\ldots,X_n$ be a random sample. Find the Cramer-Rao lower bound for the variance of unbiased estimators of θ based on the following p.d.f.

$$f\left(x,\theta\right)=\left(\frac{1}{\theta^2}\right)xe^{-x/\theta}, \quad 0<\theta<\infty, \quad 0<x<\infty$$

(b) Find a pivotal quantity based on a random sample of size n from a $N\left(\theta,\theta\right)$ population, where $\theta>0$. Use the pivotal quantity to set up a $\left(1-\alpha\right)\%$ confidence interval for θ.

(c) Let $f\left(x,\theta\right)=\left(\frac{1}{\theta}\right)x^{\left(1-\theta\right)/\theta}, \quad 0<x<1$

(i) Show that the maximum likelihood estimator of θ is $\hat{\theta}=-\left(\frac{1}{n}\right)\sum_{i=1}^{n}\ln X_i$

(ii) Show that $E\left(\hat{\theta}\right)=\theta$ and thus $\hat{\theta}$ is an unbiased estimator of θ.

[100 marks]

3. (a) *Biarkan* $X_1,\ldots,X_n$ *sebagai sampel rawak daripada populasi dengan min* μ *dan varians* σ^2. *Tunjukkan bahawa*

$$E\left[\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{\sigma}\right]=0 \quad dan \quad \operatorname{var}\left[\frac{\sqrt{n}\left(\overline{X}_n-\mu\right)}{\sigma}\right]=1$$

supaya penormalan $\overline{X}_n$ *mempunyai taburan penghad* $N(0,1)$.

(b) *Tunjukkan bahawa jika* $\sqrt{n}\left(Y_n-\mu\right)\to N\left(0,\sigma^2\right)$ *secara taburan, maka* $Y_n\to\mu$ *secara kebarangkalian.*

(c) *Biarkan* $X_1,\ldots,X_n$ *merupakan sampel rawak daripada populasi dengan f.k.k*

$$f\left(x\middle|\theta\right)=\theta x^{\theta-1},\quad 0<x<1,\quad \theta>0$$

(i) *Adakah* $\sum X_i$ *cukup untuk* θ*?*

(ii) *Cari statistik cukup lengkap untuk* θ.

[100 markah]

4. (a) *Biarkan* $X_1,\ldots,X_n$ *merupakan sampel rawak. Cari batas bawah Cramer Rao untuk varians penganggar-penganggar saksama* θ *berdasarkan f.k.k berikut*

$$f\left(x,\theta\right)=\left(\frac{1}{\theta^2}\right)xe^{-x/\theta},\quad 0<\theta<\infty,\ \ 0<x<\infty$$

(b) *Cari kuantiti pangsian berdasarkan sampel rawak bersaiz n daripada populasi* $N\left(\theta,\theta\right)$, *dimana* $\theta>0$. *Guna kuantiti pangsian untuk membina selang keyakinan* $\left(1-\alpha\right)\%$ *untuk* θ.

(c) *Biarkan* $f\left(x,\theta\right)=\left(\frac{1}{\theta}\right)x^{\left(1-\theta\right)/\theta},\quad 0<x<1$

(i) *Tunjukkan bahawa penggangar kebolehjadian maksimum bagi* θ *ialah* $\hat{\theta}=-\left(\frac{1}{n}\right)\sum_{i=1}^{n}\ln X_i$

(ii) *Tunjukkan bahawa* $E\left(\hat{\theta}\right)=\theta$ *dan dengan itu* $\hat{\theta}$ *ialah penganggar saksama bagi* θ.

[100 markah]

5. (a) Let $X_1, \ldots, X_n$ be a random sample with $N(\mu, 0.02)$ distribution. The hypothesis testing are :

$$H_0 : \mu \geq 10.34$$
$$H_1 : \mu < 10.34$$

(i) What is the critical region of size $\alpha = 0.05$ specified by the likelihood test criterion?

(ii) If $n = 50$, $\bar{X} = 10.30$, is H_0 rejected?

(iii) Now assume that the hypothesis testing are :

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta \neq \theta_0$$

we reject H_0 if $\left|\bar{X} - \theta_0\right| \Big/ \left(\sigma/\sqrt{n}\right) > c$

Find the power function of this test.

(b) Let $X_1, \ldots, X_{10}$ be independent and identically distributed with Bernoulli (p) distribution and let $Y = \sum_i X_i \sim bin(10, p)$. The hypothesis testing is $H_0 : p = \dfrac{1}{2}$ versus $H_1 : p = \dfrac{1}{4}$.

(i) Show that the ratio $f\left(y \middle| \dfrac{1}{4}\right) \Big/ f\left(y \middle| \dfrac{1}{2}\right)$ is decreasing in y.

(ii) Find the most powerful test of size $\alpha = 0.0547$ of the hypothesis that i.e. find c so that $P\left(Y \leq c \middle| p = \dfrac{1}{2}\right) = \alpha$.

[100 marks]

5. (a) *Biarkan* $X_1, \ldots, X_n$ *merupakan sampel rawak dengan taburan* $N(\mu, 0.02)$. *Ujian hipotesis ialah*

$$H_0 : \mu \geq 10.34$$
$$H_1 : \mu < 10.34$$

(i) *Apakah rantau genting bersaiz* $\alpha = 0.05$ *yang dispesifikasikan oleh kriteria ujian kebolehjadian?*

(ii) *Jika* $n = 50$, $\bar{X} = 10.30$, *adakah* H_0 *ditolak?*

(iii) *Andaikan bahawa ujian hipotesis berbentuk :*

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta \neq \theta_0$$

kita menolak H_0 *jika* $f\left(\left|\bar{X} - \theta_0\right| \big/ \left(\sigma/\sqrt{n}\right)\right) > c$

Cari fungsi kuasa ujian ini.

(b) *Biarkan* $X_1, \ldots, X_{10}$ *sebagai taburan Bernoulli* (p) *yang secaman dan tak bersandar dan* $Y = \sum_i X_i \sim bin(10, p)$. *Ujian hipotesis ialah* $H_0 : p = \frac{1}{2}$ *lawan* $H_1 : p = \frac{1}{4}$.

(i) *Tunjukkan bahawa nisbah* $f\left(y \middle| \frac{1}{4}\right) \Big/ f\left(y \middle| \frac{1}{2}\right)$ *adalah menyusut dalam* y.

(ii) *Cari ujian paling berkuasa bersaiz* $\alpha = 0.0547$ *daripada ujian hipotesis iaitu cari c supaya* $P\left(Y \leq c \middle| p = \frac{1}{2}\right) = \alpha$.

[100 markah]

APPENDIX

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