UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Consider the IBVP

$$u_t = \frac{1}{4}u_{xx}, \ 0 < x < 2, \ t > 0$$
$$u(0,t) = 0, \forall t > 0$$
$$u(2,t) = 0, \forall t > 0$$
$$u(x,0) = \sin \pi x$$

Evaluate the equation at the non-grid point $\left(x_i, t_{n+\frac{1}{2}}\right)$.

(a) Write the central difference approximation for u_t.

(b) Write the central difference approximation for u_{xx}.

(c) Write (b) in terms of an average of grid point values.

(d) Write the scheme for the IBVP. What is the name of this scheme and what can you say about it's stability as well as local truncation error.

(e) Use $\Delta t = 0.2$ and $\Delta x = 0.5$. Compute the values of u at $x = 0.5, 1, 1.5$ at $t = 0.4$ using the scheme.

[100 marks]

1. *Pertimbang masalah nilai awal-sempadan*

$$u_t = \frac{1}{4}u_{xx},\ \ 0 < x < 2,\ \ t > 0$$
$$u(0,t) = 0, \forall t > 0$$
$$u(2,t) = 0, \forall t > 0$$
$$u(x,0) = \sin \pi x$$

Nilaikan persamaan pada titik bukan grid $\left(x_i, t_{n+\frac{1}{2}}\right)$.

(a) *Tulis hampiran beza pusat untuk* u_t.

(b) *Tulis hampiran beza pusat untuk* u_{xx}.

(c) *Tulis (b) dalam sebutan purata nilai titik grid*

(d) *Tulis skema untuk masalah nilai awal-sempadan. Apa nama skema dan apa yang boleh disebut tentang kestabilan dan ralat pangkasan.*

(e) *Guna* $\Delta t = 0.2$ *dan* $\Delta x = 0.5$. *Kira nilai* u *di* $x = 0.5, 1, 1.5$ *pada* $t = 0.4$ *menggunakan skema.*

[100 markah]

2. Consider the transport equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

u is the known velocity, α is a known constant and T is the unknown transported quantity.

(a) Write the forward difference approximation for $\frac{\partial T}{\partial t}$.

(b) Write the backward difference approximation for $\frac{\partial T}{\partial x}$.

(c) Write the central difference approximation for $\frac{\partial^2 T}{\partial x^2}$.

(d) Using the above approximation, write down a scheme for the transport equation.

(e) Use the Fourier methods to determine the stability condition.

[100 marks]

2. *Pertimbang persamaan pengangkutan*

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

u *ialah halaju yang diketahui,* α *ialah pemalar diketahui dan* T *ialah kuantiti yang diangkut*

(a) *Tulis hampiran beza depan untuk* $\frac{\partial T}{\partial t}$.

(b) *Tulis hampiran beza belakang untuk* $\frac{\partial T}{\partial x}$.

(c) *Tulis hampiran beza pusat untuk* $\frac{\partial^2 T}{\partial x^2}$.

(d) *Dengan menggunakan hampiran di atas, tulis skema untuk persamaan pengangkutan.*

(e) *Guna kaedah Fourier untuk tertukan syarat kestabilan.*

[100 markah]

3. (a) The coefficient matrix of the linear equations **Ax**=**b** is defined by

$$\mathbf{A} = \begin{bmatrix} 1 & c & 1 \\ c & 1 & c \\ -c^2 & c & 1 \end{bmatrix}, \ c \text{ real and non-zero.}$$

(i) Find the range of values for c for which the Gauss-Seidel iteration converges.

(ii) Will the Jacobi iteration converge for $c=2$? Explain your answers.

(b) Consider the Poisson's equation

$$u_{xx} + u_{yy} = -10(x^2 + y^2 + 5);$$

in the domain $0 \le x, y \le 1$; subject to the conditions

$$u = 0 \ \ at \ \ x = 0, \ x = 1; \ \ u = 0 \ \ at \ \ y = 0; u = 1 \ \ at \ \ y = 1 \ \ for \ \ 0 < x < 1.$$

(i) Discretise the Poisson equation using central difference formula to both space derivatives by assuming $\Delta x = \Delta y = h = \frac{1}{3}$.

(ii) From (i), use Gauss-Seidel iteration method to find the solution of the resulting system (perform 3 iterations).

(iii) Calculate the optimal relaxation parameter ω, which maximizes the rate of convergence of the S.O.R method.

(iv) Compute the spectral radius $\rho(L_\omega)$ where L_ω is the S.O.R iteration matrix.

(c) Consider the system $A\vec{u} = \vec{b}$ where

$$A = \begin{pmatrix} 9 & 1 \\ 1 & 2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using a preconditioned system $M^{-1}A\vec{u} = M^{-1}\vec{b}$, perform two iterations values $\vec{u}^{(1)}$, $\vec{u}^{(2)}$, of the preconditioned Conjugate Gradient Method with preconditioner matrix $M = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$.

[100 marks]

3. (a) *Matriks koefisien bagi persamaan linear* $\boldsymbol{Ax}=\boldsymbol{b}$ *diberikan sebagai*

$$\boldsymbol{A} = \begin{bmatrix} 1 & c & 1 \\ c & 1 & c \\ -c^2 & c & 1 \end{bmatrix},$$ *c nombor nyata dan bukan sifar.*

(i) *Cari julat nilai bagi c di mana lelaran kaedah Gauss-Seidel adalah menumpu.*

(ii) *Adakah lelaran kaedah Jacobi akan menumpu jika c=2? Jelaskan jawapan anda dengan pengiraan.*

(b) *Pertimbangkan persamaan Poisson berikut*

$$u_{xx} + u_{yy} = -10(x^2 + y^2 + 5);$$

pada domain $0 \le x, y \le 1;$ *tertakluk pada syarat berikut*

$u = 0$ *di* $x = 0,\ x = 1;$ $u = 0$ *di* $y = 0; u = 1$ *di* $y = 1$ *untuk* $0 < x < 1.$

(i) *Dengan menganggapkan* $\Delta x = \Delta y = h = \dfrac{1}{3}$ *, diskretkan persamaan Poisson di atas menggunakan rumus beza ketengah.*

(ii) *Dari (i), gunakan lelaran Gauss-Seidel untuk selesaikan system linear berikut.(Kira sehingga 3 lelaran).*

(iii) *Kirakan parameter pengenduran optimum* ω_b *, yang memaksimumkan kadar penumpuan kaedah S.O.R.*

(iv) *Kirakan jejari spektrum* $\rho(L_\omega)$ *, dimana* L_ω *adalah matriks lelaran S.O.R.*

(c) *Pertimbangkan persamaan* $A\vec{u} = \vec{b}$ *dimana*

$$A = \begin{pmatrix} 9 & 1 \\ 1 & 2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

dengan menggunakan sistem berprasyarat $M^{-1}A\vec{u} = M^{-1}\vec{b}$ *, janakan 2 nilai lelaran* $\vec{u}^{(1)}$ *,* $\vec{u}^{(2)}$ *, bagi kaedah Kecerunan Konjugar berprasyarat dengan matriks prasyarat* $M = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}.$

[100 markah]

4. (a) Suppose A is an ***(m x m)*** symmetric, positive definite and tridiagonal matrix, whose Jacobi iterative Jacobi matrix is

$$B=(b_{ij})_{i,j=1,2,\ldots,m}=\begin{cases}\frac{1}{2} & if \quad j=i+1, \quad i=1,2,\ldots,m-1\\ 0.125 & if \quad j=i-1, \quad i=2,3.4,\ldots,m\\ 0 & if \quad j=i, \quad i=1,2,3,\ldots,m\end{cases}$$ and elsewhere.

(i) What is the spectral radius of its corresponding Gauss-Seidel iterative matrix when $m=86$?

(b) Consider the Laplace equation

$$\nabla^2 u = 0$$

on the region $0 \le x \le 1, 0 \le y \le 1$ with boundary conditions

$$u_x(0,y)=0; u(x,0)=0;$$
$$u_x(1,y)=2; u(x,1)=7;$$

(i) Discretise the above partial differential equation using the standard five-point centred difference approximation with $\Delta x = \Delta y = h = \dfrac{1}{3}$ in row-wise natural ordering and derive the resulting system $A\vec{u}=\vec{b}$ in 8 equations and 8 unknowns.

(ii) Write the Successive Over Relaxation (S.O.R) iterative formula to solve this system.

(c) Show that the Gauss Seidel iterative method in solving $A\vec{u}=\vec{b}$ is defined by

$$\vec{u}^{(k+1)} = (D-L)^{-1}U\vec{u}^{(k)} + (D-L)^{-1}\vec{b}$$

can be transformed to become of the form

$$\vec{u}^{(k+1)} = G\vec{u}^{(k)} + \vec{r}$$

where G is the Gauss Seidel iterative matrix given by

$$G = (I-F)^{-1}H$$

and

$$\vec{r} = (I-F)^{-1}\vec{g}$$

with

$$F = D^{-1}L, \ \ H = D^{-1}U, \ \ \vec{g} = D^{-1}\vec{b}, \ \ A = D-L-U .$$

[100 marks]

4. (a) *Andaikan A adalah matriks* **(m x m)** *bersifat simetri, positif tentu dan tiga pepenjuru dimana matriks lelaran Jacobi di beri sebagai*

$$B=(b_{ij})_{i,j=1,2,\ldots,m}=\begin{cases} \frac{1}{2} & if \quad j=i+1, \quad i=1,2,\ldots,m-1 \\ 0.125 & if \quad j=i-1, \quad i=2,3.4,\ldots,m \\ 0 & if \quad j=i, \quad i=1,2,3,\ldots,m \end{cases}$$ *dan dimana-mana,*

(i) *Berapakah jejari spektrum untuk lelaran matriks Gauss-Seidel sekiranya* $m=86$ *?*

(b) *Pertimbangkan persamaan Laplace berikut:*

$$\nabla^2 u = 0$$

pada rantau $0 \le x \le 1, 0 \le y \le 1$ *dengan syarat sempadan*

$$u_x(0,y)=0; u(x,0)=0;$$
$$u_x(1,y)=2; u(x,1)=7;$$

(i) *Diskretkan persamaan pembezaan di atas menggunakan anggaran beza ketengah lima-titik dengan* $\Delta x=\Delta y=h=\frac{1}{3}$ *dalam tertib baris semulajadi dan terbitkan sistem yang terhasil* $A\vec{u}=\vec{b}$ *dalam 8 persamaan dan 8 pembolehubah.*

(ii) *Tuliskan rumus lelaran Pengenduran Berlebihan Berturut-turut (S.O.R) dalam menyelesaikan masalah ini.*

(c) *Tunjukkan lelaran Gauss Seidel dalam menyelesaikan* $A\vec{u}=\vec{b}$ *yang di definisikan sebagai*

$$\vec{u}^{(k+1)} = (D-L)^{-1}U\vec{u}^{(k)} + (D-L)^{-1}\vec{b}$$

boleh di transformasikan kepada bentuk

$$\vec{u}^{(k+1)} = G\vec{u}^{(k)} + \vec{r}$$

dimana G *adalah matriks lelaran Gauss Seidel diberi oleh*

$$G=(I-F)^{-1}H$$

dan

$$\vec{r}=(I-F)^{-1}\vec{g}$$

serta

$$F=D^{-1}L, \quad H=D^{-1}U, \quad \vec{g}=D^{-1}\vec{b}, \quad A=D-L-U.$$

[100 markah]

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