UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MSS 302 – Real Analysis
[Analisis Nyata]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all eight** [8] questions.

*[**Arahan:** Jawab **semua lapan** [8] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. True/False questions.

(a) *Let* $f : [0,1] \to \mathbb{R}$ defind by $f(0) = 0$ and $f(t) = \frac{1}{\sqrt{t}}$ for $t > 0$. Then $f \in R[0,1]$ and $\int_0^1 f(t)\,dt = 2\sqrt{t}\,\Big|_0^1 = 2$.

(b) If $\{f_n\}$ is sequence of Riemann integrable functions on $[0,1]$ which converges uniformly to some function f, then f is Riemann integrable.

(c) In any metric space, an intersection of infinite numbers of open sets is an open set.

(d) If K_n is a compact subset of $\mathbb{R}$ for every n, then $\bigcup_{n=1}^{\infty} K_n$ is a compact subset of $\mathbb{R}$.

(e) Let (X,d_1) and (Y,d_2) are two metric spaces. If $f : (X,d_1) \to (Y,d_2)$ is a continuous function and $S \subset Y$ is compact, subset of X.

[5 marks]

2. (a) Prove that the Dirichlet function on $[0,1]$

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not Riemann integrable.

[5 marks]

3. Let $g : [0,1] \to \mathbb{R}$ be the zero function $g(x) = 0$ for all $x \in [0,1]$. Suppose $\{f_n\}$ is a sequence of countinuous functions which converges pointwise to g on $[0,1]$. Answer the following questions with either proof or counter example:

(a) Must the sequence $\{f_n\}$ converge uniformly to g on $[0,1]$?

(b) Consider the sequence of real numbers $\int_0^1 f_1, \int_0^1 f_2, \int_0^1 f_3, \cdots$.

Must $\lim_{n\to\infty}\left(\int_0^1 f_n\right) = 0$?

[15 marks]

1. *Soalan benar/palsu.*

(a) *Biarkan* $f : [0,1] \to \mathbb{R}$ *ditakrifkan oleh* $f(0) = 0$ *dan* $f(t) = \frac{1}{\sqrt{t}}$ *untuk* $t > 0$. *Maka* $f \in R[0,1]$ *dan* $\int_0^1 f(t)\,dt = 2\sqrt{t}\Big|_0^1 = 2$.

(b) *Jika* $\{f_n\}$ *adalah jujukan fungsi taksiran secara Riemann pada* $[0,1]$ *yang menumpu secara seragam kepada beberapa fungsi f, maka f adalah terkamirkan secara Riemann.*

(c) *Dalam sebarang ruang metrik, persilangan tak terhingga set terbuka adalah suatu set terbuka.*

(d) *Jika* K_n *ialah subset padat bagi* $\mathbb{R}$ *untuk setiap n, maka* $\bigcup_{n=1}^{\infty} K_n$ *ialah subset padat bagi* $\mathbb{R}$.

(e) *Biarkan* (X, d_1) *dan* (Y, d_2) *ialah 2 ruang metrik. Jika* $f : (X, d_1) \to (Y, d_2)$ *ialah fungsi selanjar dan* $S \subset Y$ *adalah padat, subset bagi X.*

[5 markah]

2. (a) *Buktikan bahawa fungsi Dirichlet pada* $[0,1]$

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{sebaliknya} \end{cases}$$

adalah bukan terkamirkan secara Riemann

[5 markah]

3. *Biarkan* $g : [0,1] \to \mathbb{R}$ *ialah fungsi sifar* $g(x) = 0$ *untuk semua* $x \in [0,1]$. *Andaikan* $\{f_n\}$ *ialah jujukan fungsi selanjar yang menumpu titik demi titik kepada g pada* $[0,1]$. *Jawab soalan berikut dengan bukti atau contoh lawan:*

(a) *Mestikah jujukan* $\{f_n\}$ *menumpu secara serangan kepada g pada* $[0,1]$ *?*

(b) *Pertimbangkan jujukan nombor nyata* $\int_0^1 f_1, \int_0^1 f_2, \int_0^1 f_3, \cdots$.

Mestikah $\text{had}_{n\to\infty} \left(\int_0^1 f_n\right) = 0$?

[15 markah]

4. Let (X,d) be a metric space.
   (a) What does it mean to say a sequence $\{x_n\} \subset X$ converges to $x \in X$?
   (b) State precisely what it means for $A \subset X$ to be called: an open set, a closed set, a dense set.
   (c) Prove that on open ball $B_r(x) = \{y \in X : d(y,x) < r\}$ is an open set, and a closed ball $\tilde{B}_r(x) = \{y \in X : d(y,x) \leq r\}$ is closed set.
   (d) Write the definition of a compact set in (X,d).
   (e) State what is means to say for that a metric space (X,d) is separable?

[20 marks]

5. Consider the Euclidean metric space $\mathbb{R}^n$.

   (a) Prove that every Cauchy sequence is convergent in the Euclidean space $\mathbb{R}^n$.
   (b) Show that the set $A = \{(x_1, x_2 \cdots, x_n) \in \mathbb{R}^n : x_1 = 0\}$ is a closed subset of $\mathbb{R}^n$.

[10 marks]

6. Consider the metric space $C[a,b]$ of continuous functions on $[a,b]$ with the metric defined by

$$d(f,g) = \sup\{|f(t) - g(t)| : t \in [a,b]\}.$$

   (a) Prove that sequence $\{f_n\}$ is convergent to f in the metric space $C[a,b]$ if and only if $\{f_n\}$ converges uniformly to f on $[a,b]$.
   (b) Given $f_n(x) = x^n$ on $[0,1]$. Is the sequence $\{f_n\}$ a convergent sequence in the metric space $C[0,1]$? Prove your answer.
   (c) Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{1+nx^2}$.
      (i) Show that the set $\{f_n : n \in \mathbb{N}\}$ is a bounded set in the metric space $C[0,1]$.
      (ii) Prove that the sequence $\{f_n\}$ does not contain a subsequence that converges in $C[0,1]$.

[25 marks]

4. *Biar* (X,d) *ruang metrik.*

   (a) *Apakah maksud dengan menyatakan jujukan* $\{x_n\} \subset X$ *menumpu kepada* $x \in X$ *?*

   (b) *Nyatakan dengan tepat apakah maksud* $A \subset X$ *disebut set terbuka, tertutup dan tumpat.*

   (c) *Buktikan bahawa bola terbuka* $B_r(x) = \{y \in X : d(y,x) < r\}$ *ialah set terbuka, dan bola tertutup* $\tilde{B}_r(x) = \{y \in X : d(y,x) \le r\}$ *ialah set tertutup.*

   (d) *Tulis takrifan set padat dalam* (X,d).

   (e) *Nyatakan maksud dengan menyatakan ruang metrik* (X,d) *ialah terpisahkan?*

[20 markah]

5. *Pertimbangkan ruang metrik Euclidean* $\mathbb{R}^n$.

   (a) *Buktikan bahawa setiap jujukan Cauchy adalah menumpu dalam ruang Euclidean* $\mathbb{R}^n$.

   (b) *Tunjukkan bahawa set* $A = \{(x_1, x_2 \cdots, x_n) \in \mathbb{R}^n : x_1 = 0\}$ *ialah subset tertutup pada* $\mathbb{R}^n$.

[10 markah]

6. *Pertimbangkan ruang metrik* $C[a,b]$ *bagi fungsi selanjar pada* $[a,b]$ *dengan metrik ditakrifkan oleh*

$$d(f,g) = \sup\{|f(t) - g(t)| : t \in [a,b]\}.$$

   (a) *Buktikan bahawa jujukan* $\{f_n\}$ *adalah menumpu kepada f dalam ruang metrik* $C[a,b]$ *jika dan hanya jika* $\{f_n\}$ *menumpu secara seragam kepada f pada* $[a,b]$.

   (b) *Diberi* $f_n(x) = x^n$ *pada* $[0,1]$. *Adakah jujukan* $\{f_n\}$ *suatu jujukan menumpu dalam ruang metrik* $C[0,1]$ *? Buktikan jawapan anda.*

   (c) *Biarkan* $f_n : [0,1] \to \mathbb{R}$ *ditakrifkan oleh* $f_n(x) = \frac{1}{1+nx^2}$.

      (i) *Tunjukkan bahawa set* $\{f_n : n \in \mathbb{N}\}$ *adalah set terbatas dalam ruang metrik* $C[0,1]$.

      (ii) *Buktikan bahawa jujukan* $\{f_n\}$ *tidak mengandungi suatu subjujukan yang menumpu dalam* $C[0,1]$.

[25 markah]

7. Consider the normed space ℓ^2 with the norm $\|\mathbf{a}\| = \left(\sum_{i=1}^{\infty}|a_i^2|\right)^{1/2}$ for $\mathbf{a} = (a_i)_{i\in\mathbb{N}} \in \ell^2$. Let $\mathbf{a} = \left(\frac{1}{k}\right)_{k\in\mathbb{N}}$. Find a sequence $(\mathbf{a}_n)$ in ℓ^2 such that every $\mathbf{a}_n \in \ell^2$ and every $\mathbf{a}_n \neq \mathbf{a}$ and such that $(\mathbf{a}_n)$ converges to $\mathbf{a}$ in ℓ^2. Prove your answer.

[10 marks]

8. Consider the normed space ℓ^{∞} with the norm $\|(x_n)\| = \sup\{|x_n| : n \in \mathbb{N}\}$.

Let $A = \{(x_n) \in \ell^{\infty} : x_i \neq 0 \text{ for finitely many } i\}$.

(a) Show that A is a subspace of ℓ^{∞}.

(b) Show that A is not a closed subspace of ℓ^{∞}.

[10 marks]

7. *Pertimbangkan ruang norma* ℓ^2 *dengan norma* $\|\mathsf{a}\| = \left(\sum_{i=1}^{\infty}|a_i^2|\right)^{1/2}$ *untuk* $\mathsf{a} = (a_i)_{i\in\mathbb{N}} \in \ell^2$. *Biarkan* $\mathsf{a} = \left(\frac{1}{k}\right)_{k\in\mathbb{N}}$. *Cari jujukan* (a_n) *dalam* ℓ^2 *supaya setiap* $\mathsf{a}_n \in \ell^2$ *dan setiap* $\mathsf{a}_n \neq \mathsf{a}$ *dan supaya* (a_n) *menumpu kepada* a *dalam* ℓ^2. *Buktikan jawapan anda.*

[10 markah]

8. *Pertimbangkan ruang norma* ℓ^∞ *dengan norma* $\|(x_n)\| = \sup\{|x_n| : n \in \mathbb{N}\}$.

Biarkan $A = \{(x_n) \in \ell^\infty : x_i \neq 0 \text{ untuk terhingga banyaknya } i\}$.

(a) *Tunjukkan bahawa A adalah subruang bagi* ℓ^∞.

(b) *Tunjukkan bahawa A bukan subruang tertutup bagi* ℓ^∞.

[10 markah]

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