UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

November 2010

MAT 517– Computational Linear Algebra and Function Approximation
[Aljabar Linear Pengkomputeran dan Penghampiran Fungsi]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (i) Find the least squares solution to the linear system below using the QR method based on Householder orthogonalization.

$$\begin{pmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

Give a geometrical interpretation of the solution.

(ii) Determine the singular value decomposition of

$$\mathbf{A} = \begin{pmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{pmatrix}$$

Find the pseudoinverse. Verify the solution to (i) using the pseudoinverse.

[100 marks]

2. (a) Suppose $\mathbf{A}$ can be factorized into $\mathbf{A} = \mathbf{P}^T\mathbf{LU}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix.

(i) Describe how this factorization can be used to solve $\mathbf{Ax} = \mathbf{b}$.

(ii) Count the number of floating point operations needed to compute $\mathbf{P}^T\mathbf{LU}$, given that $\mathbf{A}$ is a $n \times n$ matrix.

(b) Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Factor $\mathbf{A}$ into the form $\mathbf{A} = \mathbf{P}^T\mathbf{LU}$ as described in i). Use this factorization to solve the linaear equation:

$$\begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}.$$

[100 marks]

1. (i) *Cari penyelesaian kuasa dua terkecil kepada system linear di bawah menggunakan kaedah QR berasaskan peortogonan Householder.*

$$\begin{pmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

Berikan terjemahan geometri bagi penyelesaian tersebut.

(ii) *Tentukan peleraian nilai singular bagi*

$$\mathbf{A} = \begin{pmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{pmatrix}$$

Cari pseudo-songsangnya. Tentusahkan penyelesaian kepada (i) menggunakan pseudo-songsang.

[100 markah]

2. (a) *Katakan* $\mathbf{A}$ *boleh difaktorkan kepada* $\mathbf{A} = \mathbf{P}^T\mathbf{LU}$*, di mana* $\mathbf{P}$ *ialah matriks permutation,* $\mathbf{L}$ *ialah matriks segitiga bawah dan* $\mathbf{U}$ *ialah matriks segitiga atas.*

(i) *Huraikan bagaimana pemfaktoran ini boleh digunakan untuk menyelesaikan* $\mathbf{Ax} = \mathbf{b}$.

(ii) *Kira bilangan operasi titik terapung yang diperlukan untuk mengira* $\mathbf{P}^T\mathbf{LU}$*, diberikan bahawa* $\mathbf{A}$ *ialah matriks* $n \times n$.

(b) *Biar*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Faktorkan $\mathbf{A}$ *kepada bentuk* $\mathbf{A} = \mathbf{P}^T\mathbf{LU}$ *seperti dihuraikan dalam i). Gunakan pemfaktoran ini untuk menyelesaikan persamaan linear*

$$\begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}.$$

[100 markah]

3. (a) Given a function f defined on $[a,b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$. List down the defining conditions for the cubic spline interpolant for f.

(b) Given node points $x_0 = 0$, $x_1 = 0.05$ and $x_2 = 0.1$ in the interval $[0, 0.1]$, and, $f(x) = e^{2x}$.

(i) Find the cubic spline s with clamped boundary conditions that interpolates f.

(ii) Find an approximation for $\int_0^{0.1} e^{2x}dx$ by evaluating $\int_0^{0.1} s(x)\,dx$. Compare with the exact solution.

(iii) Determine the cubic spline S with free boundary conditions, and, compare $S(0.02)$, $s(0.02)$ and $e^{0.04} = 1.04081077$.

[100 marks]

4. Suppose $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is a set of linearly independent functions on $[a,b]$, w is a weight function for $[a,b]$, and, for $f \in C[a,b]$, a linear combination

$$P(x) = \sum_{k=0}^{n} a_k \phi_k$$

is sought to minimize the error

$$E(a_0, a_1, \ldots, a_n) = \int_a^b w(x)\left[f(x) - P(x)\right]^2 dx.$$

Let

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_j, & \text{when } j = k \end{cases}$$

(a) Show that, for each $j = 0, 1, \ldots, n$,

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x) f(x)\phi_j(x)\,dx.$$

(b) Given $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{3}$ and $\phi_3(x) = x^3 - \frac{3}{5}x$.

(i) Verify that, for each $k, j = 0, 1, 2, 3$,

$$\int_{-1}^{1} \phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_j, & \text{when } j = k \end{cases}$$

where α_j, $j = 0, 1, 2, 3$ are constants to be determined.

(ii) Henceforth, use $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ to calculate the polynomial of degree at most 3 that best approximates e^x over the interval $[-1, 1]$ in the least squares sense.

[100 marks]

3. (a) *Diberikan fungsi* f *tertakrif pada* $[a,b]$ *dan suatu set nod* $a = x_0 < x_1 < \cdots < x_n = b$. *Senaraikan syarat-syarat yang mentakrifkan interpolasi splin kubik bagi* f.

(b) *Diberikan titik nod* $x_0 = 0$, $x_1 = 0.05$ *dan* $x_2 = 0.1$ *dalam selang* $[0, 0.1]$, *dan,* $f(x) = e^{2x}$.

(i) *Cari splin kubik* s *dengan syarat sempadan ter'clamped' yang menginterpolasi* f.

(ii) *Cari suatu penghampiran kepada* $\int_0^{0.1} e^{2x}dx$ *dengan menilaikan* $\int_0^{0.1} s(x)\,dx$. *Bandingkan dengan nilai sebenar.*

(iii) *Tentukan splin kubik* S *dengan syarat sempadan bebas, dan, bandingkan* $S(0.02)$, $s(0.02)$ *dan* $e^{0.04} = 1.04081077$.

[100 markah]

4. *Katakan* $\{\phi_0, \phi_1, \ldots, \phi_n\}$ *ialah suatu set fungsi yang tidak bersansar linear pada* $[a,b]$, w *ialah fungsi pemberat pada* $[a,b]$, *dan, untuk* $f \in C[a,b]$, *satu gabungan linear*

$$P(x) = \sum_{k=0}^{n} a_k \phi_k$$

dicari untuk meminimumkan ralat

$$E(a_0, a_1, \ldots, a_n) = \int_a^b w(x)\left[f(x) - P(x)\right]^2 dx.$$

Biar

$$\int_a^b w(x)\phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & \text{bila } j \neq k \\ \alpha_j, & \text{bila } j = k \end{cases}$$

(a) *Tunjukkan bahawa, bagi setiap* $j = 0, 1, \ldots, n$,

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x) f(x)\phi_j(x)\,dx.$$

(b) *Diberi* $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{3}$ *dan* $\phi_3(x) = x^3 - \frac{3}{5}x$.

(i) *Tentusahkan bahawa, untuk setiap* $k, j = 0, 1, 2, 3$,

$$\int_{-1}^{1} \phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & \text{bila } j \neq k \\ \alpha_j, & \text{bila } j = k \end{cases}$$

di mana α_j, $j = 0, 1, 2, 3$ *adalah nilai malar yang perlu ditentukan.*

(ii) *Sehubungan itu, gunakan* $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ *untuk mengira polinomial berdarjah paling tinggi 3 yang memberikan penghampiran terbaik kepada* e^x *pada selang* $[-1, 1]$ *berdasarkan kuasa dua terkecil.*

[100 markah]

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