UNIVERSITI SAINS MALAYSIA

First Semester Examination
2010/2011 Academic Session

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MST 562 – Stochastic Processes
[Proses Stokastik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SIX pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ENAM muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all eight** [8] questions.

*[**Arahan:** Jawab **semua lapan** [8] soalan.*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. A post office is manned by clerks 1 and clerk 2. When Mr. *A* enters the post office, he finds that Mr. *B* is being served by clerk 1 and Mr. *C* is being served by clerk 2. Mr. *A* is told that he will be served as soon as either Mr. *B* or Mr. *C* leave. Suppose that the service time distribution of clerk *i* is exponential with mean $\frac{1}{\lambda_i}$, $i = 1, 2$.
   (i) What is the probability that clerk 1, serving Mr. *B*, completes service before clerk 2, serving Mr. *C*?
   (ii) What is the probability that, of the three customers, Mr. A is the last to leave the post office?

[10 marks]

2. Let S_n denote the time of the *n*-th arrival in a Poisson process $\{N(t), t \geq 0\}$ with rate λ.
   (i) Find $E[S_n]$ and $Var[S_n]$
   (ii) Find $E[S_4 \mid N(1) = 2]$.
   (iii) Determine $P\{N(t) = n + k \mid N(s) = n\}$ for $0 < s < t$.
   (iv) Find $E[N(s)N(t)]$ for $0 < s < t$,

[10 marks]

3. At all times, a box contains *N* balls, some of which are white and others are black. At each stage, a coin having probability *p*, $0 < p < 1$, of landing heads is flipped. If a head appears, then a ball is chosen from the box and is replaced with a white ball; if a tail appears then a ball is chosen from the box and is replaced with a black ball. Let X_n denote the number of white balls in the box after the *n*-th stage.
   (i) Is $\{X_n, n \geq 0\}$ a Markov chain? Explain.
   (ii) What are its classes? What are their periods? Are the classes transient or recurrent?
   (iii) Compute the transition probabilities P_{ij}.
   (iv) Let $N = 2$. Find the proportion of time in each state.

[15 marks]

4. In a branching process, let *Z* denote the number of offspring per individual in a population. If *Z* has distribution $P(Z = k) = p(1-p)^k$, for $k = 0, 1, 2, \ldots$ and $0 < p < 1$, calculate the extinction probability of this population

[10 marks]

5. Operations 1, 2 and 3 are performed in succession on a major piece of equipment. Operation *k*, where $k = 1, 2, 3$, takes a random amount of time T_k, that is exponentially distributed with parameters λ_k and all operations times are independent. Let $X(t)$ denote the operation being performed at time *t*, with time $t = 0$ marking the start of the first operation. If $\lambda_1 = 5$, $\lambda_2 = 3$ and $\lambda_3 = 13$, determine $P_n(t) = P\{X(t) = n\}$.

[10 marks]

1. *Sebuah pejabat pos dikendali oleh kerani 1 dan kerani 2. Apabila En. A masuk pejabat pos tersebut, beliau mendapati bahawa En. B sedang dilayan oleh kerani 1 dan En. C sedang dilayan oleh kerani 2. En. A diberitahu bahawa beliau akan dilayan sebaik sahaja En. B atau En. C beredar. Andaikan taburan masa layan bagi kerani i ialah taburan eksponen dengan min* $\frac{1}{\lambda_i}$, *i =1, 2.*

   (i) *Apakah kebarangkalian bahawa kerani 1 yang sedang melayan En. B, selesai melayan sebelum kerani 2 yang sedang melayan En. C?*

   (ii) *Apakah kebarangkalian bahawa antara ketiga-tiga pelanggan, En. A ialah pelanggan yang terakhir beredar dari pejabat pos tersebut?*

   [10 markah]

2. *Andaikan* S_n *mewakili masa ketibaan yang ke-n bagi suatu proses Poisson* $\{N(t), t \geq 0\}$ *dengan kadar λ.*

   (i) *Cari* $E[S_n]$ *dan* $Var[S_n]$

   (ii) *Cari* $E[S_4 \mid N(1) = 2]$.

   (iii) *Tentukan* $P\{N(t) = n + k \mid N(s) = n\}$ *bagi* $0 < s < t$.

   (iv) *Cari* $E[N(s)N(t)]$ *bagi* $0 < s < t$.

   [10 markah]

3. *Sebuah kotak sentiasa mengandungi N biji bola, yang sebilangannya berwarna putih dan selainnya hitam. Pada setiap tahap, sekeping duit syiling yang berkebarangkalian p, 0 < p < 1, untuk muncul kepala, dilambung. Sekiranya kepala muncul, maka sebiji bola dikeluarkan dari kotak tersebut dan digantikan dengan sebiji bola putih; sekiranya bunga muncul, sebiji bola dikeluarkan dan digantikan dengan sebiji bola hitam. Andaikan* X_n *mewakili bilangan bola putih di dalam kotak tersebut selepas tahap ke-n.*

   (i) *Adakah* $\{X_n, n \geq 0\}$ *suatu rantai Markov? Terangkan.*

   (ii) *Apakah kelas-kelasnya? Apakah tempohnya? Adakah kelas-kelas fana atau berulang?*

   (iii) *Hitung kebarangkalian peralihan* P_{ij}.

   (iv) *Andaikan N = 2. Cari peratusan masa dalam setiap keadaan.*

   [15 markah]

4. *Dalam suatu proses bercabang, andaikan Z mewakili bilangan anak bagi setiap individu dalam suatu populasi. Jika Z bertaburan* $P(Z = k) = p(1-p)^k$, *bagi k = 0, 1, 2, ... dan 0 < p < 1, hitung kebarangkalian kepupusan populasi ini.*

   [10 markah]

5. *Operasi 1, 2, dan 3 dijalankan secara berturutan ke atas suatu peralatan utama. Operasi k, dengan k = 1, 2, 3, memakan masa selama suatu tempoh yang rawak,* T_k, *yang bertaburan secara eksponen dengan parameter* λ_k *dan semua tempoh masa operasi adalah tak bersandar. Andaikan X(t) mewakili operasi yang dijalankan pada masa t, dengan t = 0 menandakan permulaan operasi pertama. Sekiranya* $\lambda_1 = 5$, $\lambda_2 = 3$ *dan* $\lambda_3 = 13$, *tentukan* $P_n(t) = P\{X(t) = n\}$

   [10 markah]

6. Consider a queueing system with a single server. Customers arrive according to a Poisson distribution at rate λ and join the system with probability $1/(j+1)$, where j is the number of customers already in the system. Service times are exponential with mean $1/\mu$

(i) Draw the state transition diagram with the transition rates.

(ii) Show that $P_j = \frac{\rho^j}{j!} P_0$ for $j = 0, 1, 2, \ldots$, where $\rho = \frac{\lambda}{\mu}$ and P_j is the limiting probability of j customers being in the system.

(iii) Find a formula for P_0. [Hint: $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$]

(iv) Calculate the limiting probability of there being at most 2 customers in the system if $\lambda = 10$ per day and $\mu = 5$ per day.

[20 marks]

7. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda$ for $i \geq 0$ and death rates $\mu_i = i\mu$ for $i \geq 0$. Assume that $X_0 = 0$. Let m_i be the expected time to go from state i to state i + 1.

(i) Derive the identity $m_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} m_{i-1}$, $i \geq 1$.

(ii) Determine the expected time to go from state 2 to state 5.

[15 marks]

8. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate $\lambda = 2$. A customer will only enter the bank if the server is free when he arrives. Suppose that the amount of time spent in the bank by an entering customer is a random variable having a distribution G with mean 5.

(i) What is the rate at which customers enter the bank?

(ii) What proportion of potential customers actually enter the bank?

Suppose that the amounts deposited in the bank by successive customers are independent random variables uniformly distributed over [2, 10].

(iii) What is the rate at which deposits accumulate?

[10 marks]

6. *Pertimbangkan suatu sistem giliran dengan seorang pelayan. Pelanggan tiba mengikut suatu taburan Poisson dengan kadar* λ *dan menyertai sistem dengan kebarangkalian* $1/(j+1)$ *, dengan j sebagai bilangan pelanggan yang telah berada dalam sistem. Masa layan tertabur secara eksponen dengan min* $1/\mu$.

(i) *Lukiskan gambarajah peralihan keadaan dengan kadar peralihannya.*

(ii) *Tunjukkan bahawa* $P_j = \frac{\rho^j}{j!} P_0$ *bagi j = 0,1,2,…, dengan* $\rho = \frac{\lambda}{\mu}$ *dan* P_j *ialah kebarangkalian penghad bahawa j pelanggan berada dalam sistem.*

(iii) *Dapatkan suatu rumus bagi* P_0 . *[Petua:* $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$ *]*

(iv) *Hitung kebarangkalian penghad bahawa terdapat sebanyak-banyaknya 2 pelanggan berada dalam sistem jika* $\lambda = 10$ *setiap hari dan* $\mu = 5$ *setiap hari.*

[20 markah]

7. *Pertimbangkan suatu proses kelahiran dan kematian dengan kadar kelahiran* $\lambda_i = (i+1)\lambda$ *bagi* $i \geq 0$ *dan kadar kematian* $\mu_i = i\mu$ *bagi* $i \geq 0$. *Andaikan bahawa* $X_0 = 0$. *Andaikan* m_i *sebagai masa yang dijangka untuk beralih dari keadaan i ke keadaan i + 1.*

(i) *Terbitkan identiti* $m_i = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} m_{i-1}, \quad i \geq 1.$

(ii) *Tentukan masa yang dijangka untuk untuk beralih dari keadaan 2 ke keadaan 5.*

[15 markah]

8. *Andaikan bakal-bakal pelanggan tiba di sebuah bank satu-pelayan mengikut suatu proses Poisson dengan kadar* $\lambda = 2$. *Seorang pelanggan hanya akan masuk ke dalam bank jika pelayan tersebut tidak sibuk semasa ia tiba. Andaikan amaun masa yang dihabiskan oleh seorang pelanggan yang masuk ke dalam bank ialah suatu pembolehubah rawak yang mempunyai taburan G dengan min 5.*

(i) *Apakah kadar kemasukan pelanggan ke dalam bank?*

(ii) *Apakah peratusan bakal-bakal pelanggan yang sebenarnya masuk ke dalam bank?*

Andaikan amaun wang yang disimpan di dalam bank oleh pelanggan-pelanggan berturutan ialah suatu pembolehubah rawak yang tertabur secara seragam pada [2, 10].

(iii) *Apakah kadar terkumpulnya wang simpanan?*

[10 markah]

APPENDIX

1. If X is distributed as Poisson with parameter $\lambda > 0$, then
$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0,1,2,\cdots$$

2. If X distributed as geometric with parameter $p, \quad 0 < p < 1$, then
$$P(X = x) = p(1-p)^{x-1} \quad ; \quad x = 1,2,\cdots$$

3. If X distributed as Binomial with parameter $p, \quad 0 < p < 1$, then
$$P(X = x) = \binom{n}{x} p^{x} q^{n-x} \quad ; \quad x = 0,1,2,\cdots,n$$

4. If X distributed as exponential with parameter $\lambda > 0$, then
$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

5. If X is distributed as gama with parameter $\alpha > 0$ and $\beta > 0$ then
$$f(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1} e^{-\beta x} \quad ; \quad x > 0$$

6. If X is distributed as normal with parameter μ and $\alpha^{2} > 0$ then
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^{2}/2\sigma^{2}} \quad ; \quad -\infty < x < \infty$$

7. Formula of geometric series
$$\sum_{k=0}^{\infty} ar^{k} = \frac{a}{1-r} \quad ; \quad |r| < 1$$

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