UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2010/2011

Jun 2011

MAT 517 – Computational Linear Algebra and Function Approximation
[Aljabar Linear Pengkomputeran dan Penghampian Fungsi]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Let

$$\mathbf{A} = \begin{pmatrix} 4 & -4 & 1 \\ 0 & 1 & 3 \\ -3 & 3 & -2 \end{pmatrix}.$$

(i) Use Householder transformation to compute the $\mathbf{QR}$ factorization of $\mathbf{A}$.

(ii) Consider the following linear system of equation

$$\begin{aligned} 4x_1 - 4x_2 + x_3 &= 2 \\ x_2 + 3x_3 &= 2 \\ -3x_1 + 3x_2 - 2x_3 &= 1. \end{aligned} \qquad (1)$$

Use results in part (i) to transform the linear system above into an upper triangular form. Henceforth, solve the upper triangular system.

(iii) Check your solution in part (ii) by computing the 2-norm of the residual vector of system (1).

[100 marks]

1. Biar

$$\mathbf{A} = \begin{pmatrix} 4 & -4 & 1 \\ 0 & 1 & 3 \\ -3 & 3 & -2 \end{pmatrix}.$$

(i) Guna transformasi Householder untuk mengira pemfaktoran $\mathbf{QR}$ *bagi* $\mathbf{A}$.

(ii) Pertimbangkan sistem persamaan linear berikut

$$\begin{aligned} 4x_1 - 4x_2 + x_3 &= 2 \\ x_2 + 3x_3 &= 2 \\ -3x_1 + 3x_2 - 2x_3 &= 1. \end{aligned} \qquad (1)$$

Guna keputusan di bahagian (i) untuk mentransformasi sistem linear di atas kepada bentuk segi tiga atas. Dengan yang demikian, selesaikan sistem segi tiga atas tersebut.

(iii) Semak penyelesaian anda di bahagian (ii) dengan mengira norma-2 vektor baki untuk sistem (1).

[100 markah]

2. Given

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ -1 & -2\end{pmatrix}, \text{ and } \mathbf{b}=\begin{pmatrix}6\\ -4\end{pmatrix}.$$

(i) Compute the singular value decomposition of $\mathbf{A}$.

(ii) Use your results from part (i) to show that the pseudo inverse of $\mathbf{A}$ is

$$\mathbf{A}^{+}=\frac{1}{10}\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}.$$

(iii) Use $\mathbf{A}^{+}$ to find a least squares solution to the system $\mathbf{Ax}=\mathbf{b}$.

(100 marks)

2. Diberi

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ -1 & -2\end{pmatrix}, \textit{ dan } \mathbf{b}=\begin{pmatrix}6\\ -4\end{pmatrix}.$$

(i) Kira penghuraian nilai singular $\mathbf{A}$.

(ii) Guna keputusan anda di bahagian (i) untuk menunjukkan bahawa pseudo songsang $\mathbf{A}$ *ialah*

$$\mathbf{A}^{+}=\frac{1}{10}\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}.$$

(iii) Guna $\mathbf{A}^{+}$ *untuk mencari penyelesaian kuasa dua terkecil bagi sistem* $\mathbf{Ax}=\mathbf{b}$.

[100 markah]

3. The cubic spline interpolating the data (x_i, y_i), $i=0,1,\ldots,n$ is given by

$$S(x)=\begin{cases} S_0(x) & \text{if } x_0 \le x \le x_1\\ \vdots & \vdots\\ S_{n-1}(x) & \text{if } x_{n-1} \le x \le x_n \end{cases} \qquad (2)$$

where

$$S_j(x)=a_j+b_j(x-x_j)+c_j(x-x_j)^2+d_j(x-x_j)^3, \qquad j=0,1,\ldots,n-1. \qquad (3)$$

The coefficients a_j, b_j, c_j and d_j, $j=0,1,\ldots,n-1$ are given by

$$a_j = y_j,$$
$$b_j = \frac{1}{h_j}\left(a_{j+1} - a_j\right) - \frac{h_j}{3}\left(2c_j + c_{j+1}\right),$$
$$c_j = \frac{S_j''\left(x_j\right)}{2},$$
$$d_j = \frac{1}{3h_j}\left(c_{j+1} - c_j\right), \qquad (4)$$

with $h_j = x_{j+1} - x_j$. The coefficients c_j's can be found by solving the following linear system of equation:

$$h_{j-1}c_{j-1} + 2\left(h_{j-1} + h_j\right)c_j + h_j c_{j+1} = \frac{3}{h_j}\left(a_{j+1} - a_j\right) - \frac{3}{h_{j-1}}\left(a_j - a_{j-1}\right), \qquad (5)$$

for each $j = 1, 2, \ldots, n-1$, where $c_0 = 0, c_n = 0$ (natural cubic spline).

(i) Use formulae (2), (3) and (4) to verify that the cubic spline $S(x)$ is continuous everywhere in $[x_0, x_n]$.
(ii) Outline how formula (5) are derived.
(iii) Using the data in the table below,

x_i	0	1.0	2.0	3.0
y_i	3.0	4.0	6.0	8.0

evaluate the (natural) cubic spline interpolating these data at $x = 2.4$.

[100 marks]

3. *Splin kubik yang menginterpolasi data* (x_i, y_i), $i = 0, 1, \ldots, n$ *diberi oleh*

$$S(x) = \begin{cases} S_0(x) & \text{jika } x_0 \le x \le x_1 \\ \vdots & \vdots \\ S_{n-1}(x) & \text{jika } x_{n-1} \le x \le x_n \end{cases} \qquad (2)$$

dengan

$$S_j(x) = a_j + b_j\left(x - x_j\right) + c_j\left(x - x_j\right)^2 + d_j\left(x - x_j\right)^3, \quad j = 0, 1, \ldots, n-1. \qquad (3)$$

Pekali a_j, b_j, c_j *dan* d_j, $j = 0, 1, \ldots, n-1$ *diberi oleh*

$$a_j = y_j,$$
$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}),$$
$$c_j = \frac{S_j''(x_j)}{2},$$
$$d_j = \frac{1}{3h_j}(c_{j+1} - c_j), \qquad (4)$$

dengan $h_j = x_{j+1} - x_j$. *Pekali* c_j *boleh dicari dengan menyelesaikan sistem persamaan linear berikut:*

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}), \qquad (5)$$

untuk setiap $j = 1, 2, \ldots, n-1$, *dengan* $c_0 = 0, c_n = 0$ *(splin kubik asli).*

(i) *Guna rumus (2), (3) dan (4) untuk ditentusahkan bahawa splin kubik* $S(x)$ *selanjar di mana-mana sahaja pada* $[x_0, x_n]$.

(ii) *Gariskan bagaimana rumus (5) diterbitkan.*

(iii) *Menggunakan data dalam jadual di bawah,*

x_i	*0*	*1.0*	*2.0*	*3.0*
y_i	*3.0*	*4.0*	*6.0*	*8.0*

nilaikan splin kubik (asli) yang menginterpolasi data-data tersebut pada $x = 2.4$.

[100 markah]

(4) Let $P_n(x)$ denotes the nth degree Lagrange polynomial which interpolates the data $(x_i, f(x_i))$, $i = 0,1,\ldots,n$, x_i and $y_i = f(x_i)$ are given real numbers with x_i, $i = 0,1,\ldots,n$ distinct, and,

$$P_n(x) = L_{n,0}(x) f(x_0) + L_{n,1}(x) f(x_1) + \cdots + L_{n,n}(x) f(x_n)$$
$$= \sum_{k=0}^{n} L_{n,k}(x) f(x_k)$$

where,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}.$$

for each $k = 0,1,\ldots,n$.

(i) Prove that $P_n(x)$ is indeed an interpolating polynomial to the data $(x_i, f(x_i))$, $i = 0,1,\ldots,n$.

(ii) Using the data points in the table below,

x_i	0	1	2
$y_i = e^{x_i}$	1	2.72	7.39

(a) find the 2nd degree Lagrange polynomial interpolating these data;

(b) find the

$$\max_{0\leq x\leq 2}\left|(x-x_0)(x-x_1)(x-x_2)\right|$$

and hence a bound to the error

$$\left|e^x - P_2(x)\right|, \quad 0\leq x\leq 2.$$

(HINT: Solve (for x) $\frac{d}{dx}\left[(x-x_0)(x-x_1)(x-x_2)\right]=0$).

[100 marks]

4. *Biar $P_n(x)$ mewakili polinomial Lagrange darjah n yang menginterpolasi data $(x_i, f(x_i))$, $i = 0,1,\ldots,n$, x_i dan $y_i = f(x_i)$ adalah nombor-nombor nyata yang diberi dengan x_i, $i = 0,1,\ldots,n$ berbeza, dan,*

$$P_n(x) = L_{n,0}(x)f(x_0) + L_{n,1}(x)f(x_1) + \cdots + L_{n,n}(x)f(x_n)$$

$$= \sum_{k=0}^{n} L_{n,k}(x)f(x_k)$$

dengan,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}.$$

untuk setiap $k = 0,1,\ldots,n$.

(i) *Buktikan bahawa* $P_n(x)$ *adalah sebenar-benarnya satu polinomial interpolasi kepada data* $(x_i, f(x_i))$, $i = 0,1,\ldots,n$.

(ii) *Menggunakan titik data dalam jadual di bawah,*

x_i	0	1	2
$y_i = e^{x_i}$	1	2.72	7.39

(a) *cari polinomial Lagrange darjah 2 yang menginterpolasi data tersebut;*

(b) *cari*

$$\max_{0\leq x\leq 2}\left|(x-x_0)(x-x_1)(x-x_2)\right|$$

dan dengan demikian had ralat

$$\left|e^x - P_2(x)\right|, \quad 0 \leq x \leq 2.$$

(PETUNJUK: Selesaikan (bagi x *)* $\frac{d}{dx}\left[(x-x_0)(x-x_1)(x-x_2)\right] = 0$*).*

[100 markah]

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