UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2010/2011

Jun 2011

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}.$$

You must show your calculations and not just give the answer.

(b) Consider the system

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + 3z &= 3 \\ 3x + 4y + kz &= 7 \end{aligned}$$

(i) Find all values of k for the system to have a unique solution. Show your work.

(ii) Find all values of k for the system to have an infinitely many solutions. Show your work.

(iii) Can the system be inconsistent? Explain.

(c) Given a system of linear equations with unknowns x_1, x_2, x_3 and x_4 as the following:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 &= 0 \\ -x_1 - 2x_2 - x_3 - 7x_4 &= 0 \end{aligned}$$

(i) Convert the system into augmented matrix form and obtain its reduced row-echelon form (RREF) using the Gauss-Jordan procedure. (*You must show all elementary row operations (ERO) used*)

(ii) From the RREF you obtained in (i), find the solution to the system.

(d) (i) Prove that if A is a skew-symmetric matrix then A^T is symmetric.

(ii) Let B be any lower triangular matrix with non-zero entries on the diagonal. Prove that the inverse of B exists and is also lower triangular.

[100 marks]

1. (a) Hitungkan songsangan bagi matriks

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}.$$

Anda mesti tunjukkan pengiraan anda dan bukan hanya beri jawapan.

(b) *Pertimbangkan sistem*

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + 3z &= 3 \\ 3x + 4y + kz &= 7 \end{aligned}$$

(i) *Cari semua nilai k untuk sistem tersebut mempunyai penyelesaian unik. Tunjukkan kerja anda.*

(ii) *Cari semua nilai k untuk sistem tersebut mempunyai bilangan penyelesaian yang banyak.*

(iii) *Bolehkan sistem tersebut menjadi tak konsisten? Jelaskan.*

(c) *Diberi suatu sistem persamaan linear dengan* x_1, x_2, x_3 *dan* x_4 *seperti berikut:*

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 &= 0 \\ -x_1 - 2x_2 - x_3 - 7x_4 &= 0 \end{aligned}$$

(i) *Tukar sistem berikut ke bentuk matriks imbuhan dan dapatkan bentuk eselon baris terturun (BEBT) nya menggunakan prosedur Gauss-Jordan. (Anda mesti tunjuk semua operasi baris permulaan (OBP) yang digunakan)*

(ii) *Dari BEBT yang diperoleh dalam (i), cari penyelesaian kepada sistem tersebut.*

(d) *(i)* *Buktikan bahawa jika A adalah simetri pencong maka* A^T *adalah simetri.*

(ii) *Biar B sebagai matriks segitiga bawah dengan pemasukan pepenjurunya bukan sifar. Tunjukkan bahawa songsang B juga wujud dan berbentuk segitiga bawah juga.*

[100 marks]

2. (a) Suppose that $S = A \in M_{2\times 2} \mid A = -(A^T)$.

(i) Prove that S is a subspace of $M_{2\times 2}$.

(ii) Find a basis for S.

(b) For what value(s) of h are the vectors $(1,2,1)$, $(2,5,3)$ and $(-1,-4,h)$ linearly dependent? Make sure that you justify your arguments and show all work. No marks will be given for an answer (even a correct one) if no work is shown.

(c) Let U and V be subspaces of Υ^3 where U is spanned by $1,0,-1$, $0,1,-1$ and V is spanned by $1,1,0$, $0,2,1$. Find a basis for $U \cap V$.

(d) Let $B = 1+2x, x-x^2, x+x^2$. Show that B is a basis for $P_2(\square)$.

[Note: You only need to show that B is linearly independent. However, you need to explain why this is so.]

[100 marks]

2. (a) Andai $S = \{ A \in M_{2\times 2} \mid A = -(A^T) \}$.

(i) Buktikan bahawa S adalah subruang $M_{2\times 2}$.

(ii) Find a basis for S.

(b) Apakah nilai-nilai h yang menjadikan vektor-vektor $(1,2,1)$, $(2,5,3)$ *dan* $(-1,-4,h)$ *bersandar linear? Pastikan anda justifikasikan hujah anda dan tunjukkan semua jalankerja. Tiada markah akan diberi untuk jawapan (walaupun betul) jika tiada jalankerja ditunjukkan.*

(c) Biar U dan V sebagai subruang-subruang $\mathbb{R}^3$ *yang mana U direntang oleh* $\{(1,0,-1), (0,1,-1)\}$ *dan V direntang oleh* $\{(1,1,0), (0,2,1)\}$ *. Cari asas* $U \cap V$.

(d) Biar $B = \{1+2x, x-x^2, x+x^2\}$ *. Tunjukkan bahawa B adalah asas* $P_2(\mathbb{R})$.

[Nota: Anda hanya perlu tunjukkan bahawa B adalah tak bersandar linear. Walau bagaimanapun,anda perlu jelaskan mengapa ia begitu.]

[100 marks]

3. (a) Let $A = \begin{bmatrix} 0 & 1 & 2 & 0 & -5 & 6 \\ 1 & 0 & 3 & 1 & -6 & 1 \\ 1 & 0 & 2 & 0 & -4 & 1 \end{bmatrix}$.

(i) Describe the column space of *A*, *C(A)*.

(ii) Identify the basis for *C(A)*. Explain how you obtained it.

(iii) Write each non-basis column vector c_i of *A* as a linear combination of the basis vectors obtained in (ii).

(b) Let *W* be a subspace in $\mathbb{R}^4$ of vectors (x_1, x_2, x_3, x_4) satisfying

$$2x_1 - x_3 + x_4 = 0 .$$

(i) Describe the subspace $W^{\perp}$ and determine its basis. Deduce the dimension of $W^{\perp}$. [Recall: $W^{\perp}$ is the orthogonal complement of *W*.]

(ii) Use the Gram-Schmidt process to find an orthogonal basis for *W*.

(iii) Find an orthonormal basis for *W*.

(c) Find the equation of a line *y* = *mx* + *b* that best fits the given data points: (0,1), (1,1) and (3,2). You must define all notations and show all calculations involved.

(d) (i) Suppose that **u** is orthogonal to both **v** and **w**. Show that **u** is orthogonal to any vector *r***v** + *s***w** for any scalars *r* and *s*.

(ii) Recall that if *A* is the standard matrix for a linear transformation *T*, then we can denote $\text{Ker}\, T = \text{Ker}\, A$. Show that $\text{Ker}\,(A) = \text{Ker}(A^T A)$.

[Recall: A^{T} is the transpose of *A*]

[100 marks]

3. *(a)* *Biar* $A=\begin{bmatrix} 0 & 1 & 2 & 0 & -5 & 6 \\ 1 & 0 & 3 & 1 & -6 & 1 \\ 1 & 0 & 2 & 0 & -4 & 1 \end{bmatrix}$.

(i) *Beri deskripsi ruang lajur A, C(A).*

(ii) *Kenalpasti asas untuk C(A). Jelaskan bagaimana anda memperolehnya.*

(iii) *Tulis setiap vektor lajur bukan asas* c_i *dari A sebagai gabungan linear vektor asas yang diperoleh dalam (ii).*

(b) *Biar W sebagai subruang* $\mathbb{R}^4$ *vektor-vektor* (x_1, x_2, x_3, x_4) *yang memenuhi*

$$2x_1 - x_3 + x_4 = 0.$$

(i) *Beri deskripsi subruang* $W^{\perp}$ *dan tentukan asasnya. Buat kesimpulan terhadap dimensi* $W^{\perp}$. *[Ingat:* $W^{\perp}$ *ialah pelengkap berortogan bagi W.]*

(ii) *Guna proses Gram-Schmidt untuk mencari asas berortogon bagi W.*

(iii) *Cari asas ortonormal bagi W.*

(c) *Cari persamaan garis lurus y = mx + b yang terbaik memadankan titik-titik data : (0,1), (1,1) dan (3,2). Anda mesti takrif semua tetanda dan tunjuk semua pengiraan yang terlibat.*

(d) *(i)* *Andai* ***u*** *berortogon dengan* ***v*** *dan* ***w***. *Tunjukkan bahawa* ***u*** *adalah berortogon dengan semua vektor* ***v*** + *s**w*** *untuk sebarang skalar r dan s.*

(ii) *Ingat bahawa A adalah matriks piawai bagi transformasi T, maka kita boleh tulis* $\operatorname{Ker} T = \operatorname{Ker} A$. *Tunjuk bahawa* $\operatorname{Ker}(A) = \operatorname{Ker}(A^T A)$.

[Ingat: A^T *adalah matrik transposisi bagi A]*

[100 marks]

4. (a) Given the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$(x, y, z)\,T = (x - y + 2z,\ y + 2z).$$

(i) Find the standard matrix for *T*.

(ii) Find the kernel for *T*, Ker *T*, and deduce its basis.

(iii) Find the image for *T*, Im *T*, and deduce its basis.

(iv) Is *T* one-to-one? Is *T* onto? Explain your answers.

(b) Let $E = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $B = \{b_1, b_2, b_3\}$ be another basis for $\mathbb{R}^3$. Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation with the property that

$$(x, y, z)T = (z - y)\,b_1 - (x + z)\,b_2 + (x - y)\,b_3$$

(i) Compute $(e_1)T$, $(e_2)T$ and $(e_3)T$.

(ii) Compute $\left[(e_1)T\right]_B$, $\left[(e_2)T\right]_B$ and $\left[(e_3)T\right]_B$.

(iii) Find the matrix for *T* relative to *E* and *B* (denoted as $T_{E,B}$)

(c) (i) Let B be a $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of B and if $\lambda_1 \mathbf{u} = B\mathbf{u}$ and $\lambda_2 \mathbf{v} = B\mathbf{v}$ (i.e. **u** and **v** are eigenvectors of B corresponding to λ_1 and λ_2 respectively) then prove that **u** and **v** are orthogonal. [Hint: Start by considering $\lambda_1 (\mathbf{u} \cdot \mathbf{v})$ and show that $\mathbf{u} \cdot \mathbf{v} = 0$.]

(ii) Prove that if λ is an eigenvalue of A, **w** is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$, and **w** is a corresponding eigenvector. [Note: I is the identity matrix.]

(d) Let

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

(i) Find all the eigenvalues and the eigenvectors of A.
(ii) Diagonalize A and compute A^{40}.

[100 marks]

4\. (a) *Diberi transformasi linear* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *yang ditakrifkan oleh*

$$(x, y, z)T = (x - y + 2z,\ y + 2z).$$

(i) Cari matris piawai untuk T.
(ii) Cari inti bagi T, Inti T, dan deduksikan tentang asas-asasnya.
(iii) Cari julat bagi T, Julat T, dan deduksikan tentang asas-asasnya.
(iv) Adakah T satu-ke-satu? Jelaskan jawapan anda.

(b) *Biar* $E = \{e_1, e_2, e_3\}$ *sebagai asas piawai* $\mathbb{R}^3$ *dan* $B = \{b_1, b_2, b_3\}$ *sebagai asas yang lain bagi* $\mathbb{R}^3$. *Andai* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *adalah transformasi linear yang ditakrifkan oleh*

$$(x, y, z)T = (z - y)b_1 - (x + z)b_2 + (x - y)b_3$$

(i) Kirakan $(e_1)T$, $(e_2)T$ *dan* $(e_3)T$.
(ii) Kirakan $[(e_1)T]_B$, $[(e_2)T]_B$ *dan* $[(e_3)T]_B$.
(iii) Cari matriks T relatif terhadap asas E dan B (ditandakan sebagai $T_{E,B}$*)*

(c) (i) *Biar B suatu matriks* $n \times n$ *yang simetri. Jika* λ_1 *dan* λ_2 *adalah nilai eigen B dan jika* $\lambda_1 \mathbf{u} = B\mathbf{u}$ *dan* $\lambda_2 \mathbf{v} = B\mathbf{v}$ *(i.e.* ***u*** *dan* ***v*** *adalah vektor eigen yang bersepadan dengan* λ_1 *dan* λ_2 *secara tertib) maka buktikan* ***u*** *dan* ***v*** *adalah berortogon. [Petunjuk: Mula dengan* $\lambda_1 (\mathbf{u} \cdot \mathbf{v})$ *dan tunjukkan bahawa* $\mathbf{u} \cdot \mathbf{v} = 0$.*]*

(ii) *Buktikan bahawa jika* λ *adalah nilai eigen of A,* ***w*** *ialah vektoreigen yang bersepadan, dan s suatu skalar, maka* $\lambda - s$ *ialah suatu nilai eigen* $A - sI$, *dan* ***w*** *ialah vektor eigen yang bersepadan. [Nota: I ialah matriks identiti.]*

(d) *Biar*

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

(i) *Cari semua nilai-nilai eigen dan vektor-vektor eigen A.*
(ii) *Pepenjurukan A dan kirakan* A^{40}.

[100 marks]

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