UNIVERSITI SAINS MALAYSIA

Peperiksaan Semasa Cuti Panjang
Sidang akademik 2010/2011

June 2011

MAT 202 Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **FIVE** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LIMA** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all six** [6] questions.

*[Arahan: Jawab **semua enam** [6] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1 (a) (i) Give the definitions of the supremum (least upper bound) and the infimum (greatest lower bound) of a nonempty set.
(ii) What are the supremum and the infimum of the set

$$A = \{(-1)^n(1-2^{-n}) : n \in \mathbb{N}\}?$$

(b) Show that the set $S = \{p \in \mathbb{Q} : 0 < p < \sqrt{2}\}$ has no supremum in $\mathbb{Q}$.

(c) Give the statement of the Completeness Axiom for $\mathbb{R}$.

[25 marks]

1 (a) (i) Beri takrif untuk supremum (batas atas terkecil) dan infimum (batas bawah terbesar) untuk suatu set tak kosong.
(ii) Apakah supremum dan infimum untuk set

$$A = \{(-1)^n(1-2^{-n}) : n \in \mathbb{N}\}?$$

(b) Tunjukkan bahawa set $S = \{p \in \mathbb{Q} : 0 < p < \sqrt{2}\}$ tidak mempunyai supremum dalam $\mathbb{Q}$.

(c) Beri pernyataan untuk Aksiom Kelengkapan bagi $\mathbb{R}$.

[25 markah]

2 (a) For two nonempty sets A and B, give the definition of a function f from A to B.

(b) Determine whether the following f defines a function. Give your reason.
(i) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$.
(ii) $f : \mathbb{R} \to \mathbb{N}, f(x) = 2x + 3$.
(iii) $f : \mathbb{N} \to \mathbb{N}, f(x) = 2x + 3$.

(c) (i) Given a function $f : X \to X$ with $A \subset X$ and $B \subset Y$, give the definitions of the image of A and the preimage of B under f.
(ii) Consider the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Determine the image of the closed interval $[0, 3]$ and the preimage of $[0, 9]$ under f.

[25 marks]

2 (a) Untuk dua set tak kosong A dan B, beri takrif untuk fungsi f dari A ke B.

(b) Tentukan sama ada f berikut menakrifkan satu fungsi. Beri alasan anda.
(i) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$.
(ii) $f : \mathbb{R} \to \mathbb{N}, f(x) = 2x + 3$.
(iii) $f : \mathbb{N} \to \mathbb{N}, f(x) = 2x + 3$.

(c) (i) Diberi fungsi $f : X \to X$ dengan $A \subset X$ dan $B \subset Y$, nyatakan takrif untuk imej set A dan praimej set B terhadap f.
(ii) Pertimbangkan fungsi $f : \mathbb{R} \to \mathbb{R}$ dengan $f(x) = x^2$. Tentukan imej selang tertutup $[0, 3]$ dan praimej $[0, 9]$ terhadap f.

[25 markah]

3 (a) (i) Give the $\epsilon - \delta$ definition for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point a.
(ii) Show, by using the definition in (i), that $f(x) = x^2$ is continuous at $x = 0$.

(b) (i) Give the definition of uniform continuity for a function $f : \mathbb{R} \to \mathbb{R}$.
(ii) Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x$ is uniformly continuous on $\mathbb{R}$.
(iii) Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

[25 marks]

3 (a) (i) Beri takrif $\epsilon - \delta$ untuk fungsi $f : \mathbb{R} \to \mathbb{R}$ selanjar pada titik a.
(ii) Tunjuk, dengan menggunakan takrif di bahagian (i), bahawa $f(x) = x^2$ selanjar pada $x = 0$.

(b) (i) Beri takrif keselanjaran secara seragam untuk suatu fungsi $f : \mathbb{R} \to \mathbb{R}$.
(ii) Tunjukkan bahawa $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x$ selanjar secara seragam pada $\mathbb{R}$.
(iii) Tunjukkan bahawa $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ tak selanjar secara seragam pada $\mathbb{R}$.

[25 markah]

4 (a) The limit $\lim_{n\to\infty} a_n = a$ means that for any $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ for all $n \geq N$. Show that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

(b) Find the limit $\lim_{n\to\infty} |a_n|$, where $a_n = (-1)^n$.
Does the limit $\lim_{n\to\infty} a_n$ exist?

(c) If $\{a_n\}$ is a sequence that converges to a, show that any of its subsequences $\{a_{n_k}\}$ also converges to a.

[25 marks]

4 (a) Had $\lim_{n\to\infty} a_n = a$ *bermaksud untuk setiap* $\epsilon > 0$, *wujud nombor tabii* N *supaya* $|a_n - a| < \epsilon$ *untuk setiap* $n \geq N$. *Tunjukkan bahawa* $\lim_{n\to\infty} a_n = 0$ *jika dan hanya jika* $\lim_{n\to\infty} |a_n| = 0$.

(b) Cari had $\lim_{n\to\infty} |a_n|$, *dengan* $a_n = (-1)^n$.
Adakah had $\lim_{n\to\infty} a_n$ *wujud?*

(c) Jika jujukan $\{a_n\}$ *menumpu ke* a, *tunjukkan bahawa setiap subjujukannya* $\{a_{n_k}\}$ *juga menumpu ke* a.

[25 markah]

5 (a) (i) Let $f_n(x) = \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Find the limit function for the sequence $\{f_n\}$.
(ii) Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$.

(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges uniformly on $\mathbb{R}$.

(c) (i) If $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$, then is f continuous on $\mathbb{R}$? Give your reason.
(ii) Show that $\int_0^1 f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n} \tan^{-1}\left(\frac{1}{n}\right)$.

[25 marks]

5 (a) *(i) Andaikan* $f_n(x) = \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. *Cari had fungsi untuk jujukan* $\{f_n\}$.
(ii) Tunjukkan bahawa $\{f_n\}$ *menumpu secara seragam pada* $\mathbb{R}$.

(b) *Tunjukkan bahawa* $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ *menumpu secara seragam pada* $\mathbb{R}$.

(c) *(i) Jika* $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$, *maka adakah* f *selanjar pada* $\mathbb{R}$? *Beri alasan anda.*
(ii) Tunjukkan bahawa $\int_0^1 f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n} \tan^{-1}\left(\frac{1}{n}\right)$.

[25 markah]

6 (a) For each of the following sets, find its interior points, limit points and isolated points.
(i) $X = [1,5) - \mathbb{Q}$.
(ii) $Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(b) Determine whether each of the following sets is open, closed or neither. Which one is compact?
(i) $A = \{a\}$, $a \in \mathbb{R}$.
(ii) $B = \bigcup_{n=1}^{\infty} (-n, n)$.

(c) Let A and B be subsets of $\mathbb{R}$.
(i) Show that $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.
(ii) Give an example to show that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

[25 marks]

6 (a) *Untuk setiap set berikut, cari titik pedalaman, titik had dan titik terpencil.*
(i) $X = [1,5) - \mathbb{Q}$.
(ii) $Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(b) *Tentukan sama ada setiap set berikut adalah terbuka, tertutup atau bukan. Yang mana satu adalah padat?*
(i) $A = \{a\}$, $a \in \mathbb{R}$.
(ii) $B = \bigcup_{n=1}^{\infty} (-n, n)$.

(c) *Andaikan* A *dan* B *subset kepada* $\mathbb{R}$.
(i) Tunjukkan bahawa $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.
(ii) Beri satu contoh yang menunjukkan $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

[25 markah]

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