UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

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MST 565 – Linear Model
[Model Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Consider the symmetric nonsingular matrix $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}'_{12} & a_{22} \end{pmatrix}$ in which $\mathbf{A}_{11}$ is a square matrix, a_{22} is a 1×1 matrix, and $\mathbf{a}_{12}$ is a vector. Then, if $\mathbf{A}_{11}^{-1}$ exists, show that the inverse of $\mathbf{A}$ is given as:

$$\mathbf{A}^{-1} = \frac{1}{b}\begin{pmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1} a'_{12} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1} a_{12} \\ -a'_{12}\mathbf{A}_{11}^{-1} & 1 \end{pmatrix}.$$

where $b = a_{22} - \mathbf{a}'_{12}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$.

[5 marks]

(b) Let $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$. Find the two generalized inverses of $\mathbf{A}$, denoted as $\mathbf{A}_1^-$ and $\mathbf{A}_2^-$ and verify that $\mathbf{A}\mathbf{A}_1^-\mathbf{A} = \mathbf{A}$ and $\mathbf{A}\mathbf{A}_2^-\mathbf{A} = \mathbf{A}$.

[5 marks]

(c) Using matrix $\mathbf{A}$ in (b) and consider $\mathbf{A}_1^- = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, state the relationships between $\mathrm{tr}\ \mathbf{A}^-\mathbf{A}$, $\mathrm{tr}\ \mathbf{A}\mathbf{A}^-$ and $\mathrm{rank}\ \mathbf{A}$.

[5 marks]

(d) If $\mathbf{A}$ is any 3×3 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = 5$ and $\lambda_3 = -3$, determine $|\mathbf{A}|$ and $\mathrm{tr}\ \mathbf{A}$.

[5 marks]

1. (a) *Pertimbangkan satu matriks tak singular yang simetri* $\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{a}_{12}\\ \mathbf{a}'_{12} & a_{22}\end{pmatrix}$, *yang mana* $\mathbf{A}_{11}$ *matriks segi-empat,* a_{22} *matriks 1 x 1, dan* $\mathbf{a}_{12}$ *suatu vektor. Seterusnya, jika wujud* $\mathbf{A}_{11}^{-1}$*,tunjukkan bahawa songsangan bagi* $\mathbf{A}$ *diberikan sebagai*

$$\mathbf{A}^{-1}=\frac{1}{b}\begin{pmatrix} b\mathbf{A}_{11}^{-1}+\mathbf{A}_{11}^{-1}a'_{12}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}a_{12}\\ -a'_{12}\mathbf{A}_{11}^{-1} & 1\end{pmatrix}$$

yang mana $b=a_{22}-\mathbf{a}'_{12}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$.

[5 markah]

(b) *Biar* $\mathbf{A}=\begin{pmatrix}2 & 2 & 3\\ 1 & 0 & 1\\ 3 & 2 & 4\end{pmatrix}$. *Cari dua songsangan teritlak bagi* $\mathbf{A}$, *yang diwakili sebagai* $\mathbf{A}_1^-$ *dan* $\mathbf{A}_2^-$ *dan tentusahkan bahawa* $\mathbf{A}\mathbf{A}_1^-\mathbf{A}=\mathbf{A}$ *dan* $\mathbf{A}\mathbf{A}_2^-\mathbf{A}=\mathbf{A}$.

[5 markah]

(c) *Menggunakan matriks* $\mathbf{A}$ *dalam (b) dan pertimbangkan* $\mathbf{A}_1^-=\begin{pmatrix}0 & 1 & 0\\ \frac{1}{2} & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$,

nyatakan hubungan antara $\textit{surihan}\ \mathbf{A}^-\mathbf{A}$, $\text{surihan}\ \mathbf{A}\mathbf{A}^-$ *dan* $\text{pangkat}\ \mathbf{A}$.

[5 markah]

(d) *Jika* $\mathbf{A}$ *suatu matriks* 3×3 *dengan nilai eigen* $\lambda_1=1, \lambda_2=5$ *dan* $\lambda_3=-3$, *tentukan* $|\mathbf{A}|$ *dan* $\textit{surihan}\ \mathbf{A}$.

[5 markah]

2. (a) Let $\mathbf{y} = (y_1, y_2, y_3)'$ be a random vector with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}$$

(i) Let $z = 2y_1 - 3y_2 + y_3$. Find $E(z)$ and $Var(z)$

(ii) Let $z_1 = y_1 + y_2 + y_3$ and $z_2 = 3y_1 + y_2 - 2y_3$. Find $E(\mathbf{z})$ and $Var(\mathbf{z})$ where $\mathbf{z} = (z_1, z_2)'$.

[8 marks]

(b) Let the random vector $\mathbf{v}$ be $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}.$$

If $\mathbf{v}$ is partitioned as $\mathbf{v} = (y_1, y_2, x_1, x_2)'$

(i) Determine $\boldsymbol{\mu}_y, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{yy}, \boldsymbol{\Sigma}_{yx}$ and $\boldsymbol{\Sigma}_{xx}$.

(ii) Find the conditional distribution of $\mathbf{y}|\mathbf{x}$.

(iii) Find matrix of partial correlations, $\mathbf{P}_{12.34}$

[16 marks]

(c) Let $\mathbf{v} = \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = (y_1, y_2, x_1, x_2)'$ be a partitioned random vector with mean vector and covariance matrix given as in Question 2(b). Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, a matrix of constants, determine the expected value of $\mathbf{x}'\mathbf{A}\mathbf{y}$.

[6 marks]

2. (a) *Biar* $\mathbf{y} = (y_1, y_2, y_3)'$ *sebagai satu vektor rawak dengan vektor min dan matriks kovarians*

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}$$

(i) *Biar* $z = 2y_1 - 3y_2 + y_3$. *Cari* $E(z)$ *dan* $Var(z)$

(ii) *Biar* $z_1 = y_1 + y_2 + y_3$ *dan* $z_2 = 3y_1 + y_2 - 2y_3$. *Cari* $E(\mathbf{z})$ *dan* $Var(\mathbf{z})$, *yang mana* $\mathbf{z} = (z_1, z_2)'$.

[8 markah]

(b) *Biar vektor rawak* $\mathbf{v}$ *sebagai* $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *dengan*

$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}.$$

Jika $\mathbf{v}$ *terpetak sebagai* $\mathbf{v} = (y_1, y_2, x_1, x_2)'$

(i) *Tentukan* $\boldsymbol{\mu}_y, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{yy}, \boldsymbol{\Sigma}_{yx}$ *dan* $\boldsymbol{\Sigma}_{xx}$.

(ii) *Cari taburan bersyarat untuk* $\mathbf{y}|\mathbf{x}$.

(iii) *Cari matriks korelasi separa,* $\mathbf{P}_{12.34}$.

[16 markah]

(c) *Biar* $\mathbf{v} = \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = (y_1, y_2, x_1, x_2)'$ *satu vektor rawak terpetak dengan vektor min dan matriks kovarians seperti diberikan dalam Soalan 2(b). Diberikan satu matriks pemalar,* $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, *tentukan nilai jangkaan bagi* $\mathbf{x}'\mathbf{A}\mathbf{y}$.

[6 markah]

3. (a) Suppose that $\mathbf{y}$ is $N_3\ \boldsymbol{\mu},\sigma^2\mathbf{I}$ and let

$$\boldsymbol{\mu}=\begin{pmatrix}3\\-2\\1\end{pmatrix},\ \mathbf{A}=\frac{1}{3}\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}1&1&1\\1&0&-1\end{pmatrix}$$

(i) Determine the distribution of $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma^2$.

(ii) Verify either $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{B}\mathbf{y}$ are independent or not?

(iii) What about $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $y_1+y_2+y_3$, are they independent or not?

[12 marks]

(b) Assuming that $\mathbf{y}$ is $N_4\ \boldsymbol{\mu},\boldsymbol{\Sigma}$, where

$$\boldsymbol{\mu}=\begin{pmatrix}3\\-2\\1\\4\end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&3&-4\\0&0&-4&6\end{pmatrix}.$$

Find a matrix $\mathbf{A}$ such that $\mathbf{y}'\mathbf{A}\mathbf{y}$ is $\chi^2\left(4,\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}\right)$ then determine the value of λ.

[6 marks]

(c) Suppose that for a particular linear model, we found that

$$\mathbf{X}'\mathbf{X}=\begin{pmatrix}2&1&1\\1&1&0\\1&0&1\end{pmatrix} \text{ and } \mathbf{X}'\mathbf{y}=\begin{pmatrix}14\\6\\8\end{pmatrix}$$

and the normal equations are

$$\begin{pmatrix}2&1&1\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}b_0\\b_1\\b_2\end{pmatrix}=\begin{pmatrix}14\\6\\8\end{pmatrix}.$$

(i) Determine either the model is a full- rank or not.

(ii) Using $M_1=\begin{pmatrix}2&1\\1&1\end{pmatrix}$, find the generalized inverse of $\mathbf{X}'\mathbf{X}^{-}$.

(iii) Check either β_0 is estimable or not

[7 marks]

3. (a) *Katakan bahawa* $\mathbf{y}$ *adalah* $N_3(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ *dan biar*

$$\boldsymbol{\mu} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(i) *Tentukan taburan bagi* $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma^2$.

(ii) *Sahkan sama ada* $\mathbf{y}'\mathbf{A}\mathbf{y}$ *dan* $\mathbf{B}\mathbf{y}$ *adalah tak bersandar atau tidak?*

(iii) *Bagaimana dengan* $\mathbf{y}'\mathbf{A}\mathbf{y}$ *dan* $y_1 + y_2 + y_3$, *adakah ia tak bersandar atau tidak?*

[12 markah]

(b) *Andaikan bahawa* $\mathbf{y}$ *sebagai* $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, *yang mana*

$$\boldsymbol{\mu} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 4 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -4 & 6 \end{pmatrix}.$$

Cari satu matriks $\mathbf{A}$ *supaya* $\mathbf{y}'\mathbf{A}\mathbf{y}$ *merupakan* $\chi^2\left(4, \frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}\right)$, *kemudian tentukan nilai* λ.

[6 markah]

(c) *Katakan bahawa untuk satu model linear, kita dapati bahawa*

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ dan } \mathbf{X}'\mathbf{y} = \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}$$

dan persamaan normal sebagai

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \\ 8 \end{pmatrix}.$$

(i) *Tentukan sama ada model tersebut berpangkat penuh atau tidak.*

(ii) *Menggunakan* $M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, *dapatkan songsangan teritlak* $(\mathbf{X}'\mathbf{X})^-$.

(iii) *Semak sama ada* β_0 *teranggarkan atau tidak?*

[7 markah]

4. (a) Consider the one-way classification model
$y_{ij} = \mu + \tau_i + \varepsilon_{ij}; i = 1, 2, j = 1, 2$

(i) Write the model in matrix form $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ and determine the rank of $\mathbf{X}$.

(ii) Identify the estimable functions, $\boldsymbol{\gamma} = \mathbf{U}\boldsymbol{\beta}$.

(iii) Determine $\mathbf{Z}$ by using a reparameterize technique to reduce the non-full rank model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ to a full- rank model $\mathbf{y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

[15 marks]

(b) Weight gains in cows subjected to three different treatments are given in Table 1.

Table 1: Weight Gain of Cows Subjected to Three Treatments

Treatment		
1	2	3
34.6	38.8	26.7
35.1	39.0	26.7
35.3	40.1	27.0
35.8	40.9	27.1
36.1	41.0	27.5
36.5	43.2	28.1
36.8	44.9	28.1
37.2	46.9	28.7
37.4	51.6	30.7
37.7	53.6	31.2

Test the hypothesis of equal mean treatment effects.

[10 marks]

4. (a) *Pertimbangkan model sehala*

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij};\ i = 1, 2,\ j = 1, 2$$

(i) *Tuliskan model tersebut dalam bentuk matriks* $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ *dan tentukan pangkat* $\mathbf{X}$.

(ii) *Kenalpasti fungsi teranggarkan* $\boldsymbol{\gamma} = \mathbf{U}\boldsymbol{\beta}$.

(iii) *Tentukan* $\mathbf{Z}$ *menggunakan teknik parameterkan untuk turunkan model tak berpangkat penuh* $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ *kepada model berpangkat penuh* $\mathbf{y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

[15 markah]

(b) *Pertambahan berat pada lembu bergantung kepada tiga rawatan berlainan diberikan dalam Jadual 1.*

Jadual 1*: Pertambahan Berat Pada Lembu Bergantung Kepada Tiga Rawatan*

Rawatan		
1	*2*	*3*
34.6	*38.8*	*26.7*
35.1	*39.0*	*26.7*
35.3	*40.1*	*27.0*
35.8	*40.9*	*27.1*
36.1	*41.0*	*27.5*
36.5	*43.2*	*28.1*
36.8	*44.9*	*28.1*
37.2	*46.9*	*28.7*
37.4	*51.6*	*30.7*
37.7	*53.6*	*31.2*

Uji hipotesis kesamaan min kesan rawatan.

[10 markah]

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