UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

April/May 2010

MSS 301 – Complex Analysis
[Analisis Kompleks]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all thirteen** [13] questions.

*[**Arahan:** Jawab **semua tiga belas** [13] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Define a complex number, and the operations of addition and multiplication of complex numbers. Show that every complex number can be expressed in its Cartesian form $z=a+ib$, where i is the unit imaginary number.

[20/250]

2. Solve each equation, and express the solution in its Cartesian form:

   (a) $z^3=-8i$

   (b) $\cos z=2i$

[20/250]

3. Show that the angle between two vectors z and w in the complex plane is given by $\arg(z/w)$. Use this fact to show that if α, β and γ are vertices of an equilateral triangle, then

$$\arg\left(\frac{\beta-\alpha}{\gamma-\alpha}\right)=\frac{\pi}{3},$$

and thus $\dfrac{\beta-\alpha}{\gamma-\alpha}=e^{i\pi/3}$.

Consequently show that $(\beta-\alpha)(\beta-\gamma)=(\gamma-\alpha)(\alpha-\gamma)$. Thus deduce that if α, β and γ are vertices of an equilateral triangle, then

$$\alpha^2+\beta^2+\gamma^2=\alpha\beta+\beta\gamma+\alpha\gamma.$$

[20/250]

4. If $f(z)=u(x,y)+iv(x,y)$ is differentiable at z_0, show that the Cauchy-Riemann equations hold at $z_0=x_0+iy_0$:

$$u_x(x_0,y_0)=v_y(x_0,y_0)\ ;\ u_y(x_0,y_0)=-v_x(x_0,y_0).$$

Deduce that $f'(z_0)=\dfrac{\partial f}{\partial x}(z_0)=-i\dfrac{\partial f}{\partial y}(z_0)$.

[20/250]

1. *Define a complex number, and the operations of addition and multiplication of complex numbers. Show that every complex number can be expressed in its Cartesian form* $z = a + ib$, *where* i *is the unit imaginary number.*

[20/250]

2. *Solve each equation, and express the solution in its Cartesian form:*

   (a) $z^3 = -8i$

   (b) $\cos z = 2i$

[20/250]

3. *Show that the angle between two vectors* z *and* w *in the complex plane is given by* $\arg(z/w)$. *Use this fact to show that* α, β *and* γ *are vertices of an equilateral triangle, then*

$$\arg\left(\frac{\beta-\alpha}{\gamma-\alpha}\right) = \frac{\pi}{3},$$

And thus $\dfrac{\beta-\alpha}{\gamma-\alpha} = e^{i\pi/3}$.

Consequently show that $(\beta-\alpha)(\beta-\gamma) = (\gamma-\alpha)(\alpha-\gamma)$. *Thus deduce that if* α, β *and* γ *are vertices of on equilateral triangle, then*

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \alpha\gamma.$$

[20/250]

4. *If* $f(z) = u(x,y) + iv(x,y)$ *is differentiable at* z_0, *show that the Cauchy-Riemann equations hold at* $z_0 = x_0 + iy_0$:

$$u_x(x_0,y_0) = v_y(x_0,y_0)\ ;\ u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Deduce that $f'(z_0) = \dfrac{\partial f}{\partial x}(z_0) = -i\dfrac{\partial f}{\partial y}(z_0)$.

[20/250]

5. Determine where $f(z)=f(x+iy)=x^3+iy^3-2i$ is differentiable, and find its derivative. Where is f analytic?

[20/250]

6. If $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, show that both u and v are harmonic in D. If $u(x,y)=2x(2-y)$, find its harmonic conjugate.

[20/250]

7. (a) What is the image of the horizontal line $y=b$ under the mapping $w=e^z$?

   (b) Explain why the principal branch of the logarithm, $f(z)=\mathrm{Log}\, z=\ln r+i\theta$ in $D=\{z=re^{i\theta}:r>0,-\pi<\theta\leq\pi\}$ is not differentiable? Where is $w=\mathrm{Log}\, z$ analytic?

   (c) Show that $w=\cos z$ is unbounded in the complex plane.

[20/250]

8. Let γ be a curve in the complex plane, and f be continuous over γ. Define $\int_\gamma f(z)\,dz$.

[10/250]

9. If γ is the circular arc $\gamma(t)=z_0+re^{it}$, $a\leq t\leq b$, show that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}=\frac{b-a}{2\pi}.$$

Deduce that the value of the expression is an integer if and only if γ is closed.

[15/250]

5. *Determine where* $f(z)=f(x+iy)=x^3+iy^3-2i$ *is differentiable, and find its derivative. Where is* f *analytic?*

[20/250]

6. *If* $f(z)=u(x,y)+iv(x,y)$ *is analytic in a domain* D, *show that both* u *and* v *are harmonic in* D. *If* $u(x,y)=2x(2-y)$, *find its harmonic conjugate.*

[20/250]

7. (a) *What is the image of the horizontal line* $y=b$ *under the mapping* $w=e^z$?

   (b) *Explain why the principal branch of the logarithm,* $f(z)=\mathrm{Log}\,z=\ln r+i\theta$ *in* $D=\{z=re^{i\theta}:r>0,-\pi<\theta\leq\pi\}$ *is not differentiable? Where is* $w=\mathrm{Log}\,z$ *analytic?*

   (c) *Show that* $w=\cos z$ *is unbounded in the complex plane.*

[20/250]

8. *Let* γ *be a curve in the complex plane, and* f *be continuous over* γ. *Define* $\int_\gamma f(z)\,dz$.

[10/250]

9. *If* γ *is the circular arc* $\gamma(t)=z_0+re^{it}$, $a\leq t\leq b$, *show that*

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}=\frac{b-a}{2\pi}.$$

Deduce that the value of the expression is an integer if and only if γ *is closed.*

[15/250]

10. Evaluate the following integrals over the given positively oriented simple closed contour:

(a) $\oint_{|z|=2} \frac{ze^z}{2z-3}\,dz$ (b) $\oint_{|z|=3} \frac{e^{2z}}{(z-2)^{n+1}}\,dz$

(c) $\oint_{|z|=2} \frac{e^z}{z(z+i)}\,dz$

[30/250]

11. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $|b|<|a|$.

[15/250]

12. Find three Laurent series expansion for the function

$$f(z) = \frac{2z-1}{z^2-z-6}$$

in powers of z.

[25/250]

13. If f is an entire function satisfying $|f(z)| \leq Me^x$, $z = x+iy, M > 0$, show that there exists a constant α satisfying $|\alpha| \leq M$ so that $f(z) = \alpha e^z$.

(Hint: consider the function $g(z) = f(z)e^{-z}$.)

[15/250]

10. *Evaluate the following integrals over the given positively oriented simple closed contour:*

(a) $\oint_{|z|=2} \frac{ze^z}{2z-3}\,dz$ (b) $\oint_{|z|=3} \frac{e^{2z}}{(z-2)^{n+1}}\,dz$

(c) $\oint_{|z|=2} \frac{e^z}{z(z+i)}\,dz$

[30/250]

11. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $|b|<|a|$.

[15/250]

12. *Find three Laurent series expansion for the function*

$$f(z)=\frac{2z-1}{z^2-z-6}$$

in power of z.

[25/250]

13. *If* f *is an entire function satisfying* $|f(z)|\le Me^x, z=x+iy, M>0,$ *show that there exists a constant* α *satisfying* $|\alpha|\le M$ *so that* $f(z)=\alpha e^z$.

(Hint: consider the function $g(z)=f(z)e^{-z}$.*)*

[15/250]

- ooo O ooo -