UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

April/May 2010

MGM 563 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of <u>TEN</u> pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi <u>SEPULUH</u> muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

<u>Instructions</u> : Answer **<u>all four</u>** [4] questions.

*[**<u>Arahan</u>** : Jawab **<u>semua empat</u>** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) A box contains 8 balls, numbered 1, 1, 1, 1, 2, 2, 3 and 4. A ball is drawn from the box. Let X denote the number of the ball drawn. Find
   (i) the moment generating function (mgf) of X.
   (ii) the expectation of X, $E(X)$.
   (iii) the variance of X, $\text{Var}(X)$.

[30 marks]

(b) Let X be a random variable with probability density function (pdf)

$$f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}.$$

Find the pdf for $Y = X^2$ using the distribution function method.

[30 marks]

(c) Let $f_{X_1|X_2}(x_1|x_2) = \frac{c_1 x_1}{x_2^2}$, $0 < x_1 < x_2$, $0 < x_2 < 1$, zero elsewhere denotes the conditional pdf of X_1, given $X_2 = x_2$. Also, let $f_{X_2}(x_2) = c_2 x_2^4$, $0 < x_2 < 1$, zero elsewhere denotes the marginal pdf of X_2. Find
   (i) the constants c_1 and c_2.
   (ii) the joint pdf of X_1 and X_2.
   (iii) $P\left(\frac{1}{4} < X_1 < \frac{1}{2} \,\middle|\, X_2 = \frac{5}{8}\right)$.
   (iv) $P\left(\frac{1}{4} < X_1 < \frac{1}{2}\right)$.

[40 marks]

1. (*a*) *Suatu kotak mengandungi 8 bola bernombor 1, 1, 1, 1, 2, 2, 3 dan 4. Satu bola dikeluarkan daripada kotak itu. Biarkan X mewakili nombor bola yang dikeluarkan. Cari*
   - *(i) fungsi penjana momen (fpm) untuk X.*
   - *(ii) jangkaan X, E(X).*
   - *(iii) varians X, Var(X).*

[30 markah]

(b) *Biarkan X sebagai pembolehubah rawak dengan fungsi ketumpatan kebarangkalian (fkk)*

$$f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{sebaliknya} \end{cases}.$$

Cari fkk untuk $Y = X^2$ *dengan menggunakan kaedah fungsi taburan.*

[30 markah]

(c) *Biarkan* $f_{X_1|X_2}(x_1|x_2) = \frac{c_1 x_1}{x_2^2}$, $0 < x_1 < x_2$, $0 < x_2 < 1$, *sifar di tempat lain mewakili fkk bersyarat* X_1, *diberi* $X_2 = x_2$. *Juga biarkan* $f_{X_2}(x_2) = c_2 x_2^4$, $0 < x_2 < 1$, *sifar di tempat lain mewakili fkk sut untuk* X_2. *Cari*
   - *(i) pemalar-pemalar* c_1 *dan* c_2.
   - *(ii) fkk tercantum* X_1 *dan* X_2.
   - *(iii)* $P\left(\frac{1}{4} < X_1 < \frac{1}{2} \,\middle|\, X_2 = \frac{5}{8}\right)$.
   - *(iv)* $P\left(\frac{1}{4} < X_1 < \frac{1}{2}\right)$.

[40 markah]

2. (a) Assume that X_1 and X_2 are random variables having a $N(0,1)$ distribution from a sample of size 2, while Y_1 and Y_2 are random variables with a $N(0,3)$ distribution from another sample, also of size 2. If both samples are independent, find the distribution of the following statistics:

(i) $\bar{X}+\bar{Y}$

(ii) $\frac{1}{6}(Y_1-Y_2)^2$

(iii) $\frac{\sqrt{3}(\bar{X}+\bar{Y})}{Y_1-Y_2}$

[30 marks]

(b) Let the random variable Y_n have the binomial, $b(n, p)$ distribution.

(i) Prove that $\frac{Y_n}{n}$ converges in probability to p.

(ii) Does $1-\frac{Y_n}{n}$ converge in probability to $1 - p$? Explain.

[30 marks]

(c) If $Y_1 < Y_2 < Y_3$ represent the order statistics for a random sample of size three from a $U(0, 1)$ distribution, show that Y_1/Y_2, Y_2/Y_3 and Y_3 are stochastically independent.

[30 marks]

(d) Find the method of moments estimate for λ if a random sample of size n is taken from the exponential pdf, $f_Y(y)=\lambda e^{-\lambda y}$, $y \geq 0$.

[10 marks]

2. (a) *Andaikan bahawa X_1 dan X_2 adalah pembolehubah rawak dengan taburan N(0,1) daripada sampel saiz 2, manakala Y_1 dan Y_2 adalah pembolehubah rawak dengan taburan N(0,3) daripada sampel lain, juga dengan saiz 2. Jika kedua-dua sampel adalah tak bersandar, cari taburan untuk statistik-statistik berikut:*

   (i) $\bar{X}+\bar{Y}$

   (ii) $\frac{1}{6}\ Y_1-Y_2\ ^{2}$

   (iii) $\dfrac{\sqrt{3}\ \bar{X}+\bar{Y}}{Y_1-Y_2}$

[30 markah]

(b) *Biarkan pembolehubah rawak Y_n mempunyai taburan binomial, b(n, p).*

   (i) *Buktikan bahawa $\frac{Y_n}{n}$ menumpu secara kebarangkalian kepada p.*

   (ii) *Adakah $1-\frac{Y_n}{n}$ menumpu secara kebarangkalian kepada 1 – p? Jelaskan.*

[30 markah]

(c) *Jika $Y_1 < Y_2 < Y_3$ mewakili statistik tertib untuk suatu sampel rawak saiz tiga daripada taburan U(0, 1), tunjukkan bahawa Y_1/Y_2, Y_2/Y_3 dan Y_3 adalah tak bersandar secara stokastik.*

[30 markah]

(d) *Cari anggaran kaedah momen untuk λ jika suatu sampel rawak saiz n diambil daripada fkk eksponen, $f_Y(y)=\lambda e^{-\lambda y}$, $y \geq 0$.*

[10 markah]

3. (a) A random sample of size 8 consisting of $x_1=1,\ x_2=0,\ x_3=1,\ x_4=1,\ x_5=0,\ x_6=1,$ $x_7=1$ and $x_8=0$ is taken from a population with probability mass function (pmf)

$$p_X(x;\theta)=\theta^x(1-\theta)^{1-x},\ x = 0, 1;\ 0 < \theta < 1.$$

Find the maximum likelihood estimate for θ.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from the uniform distribution with pdf $f_X(x;\theta)=\frac{1}{\theta},\ 0 \le x \le \theta.$

(i) Find the Cramer-Rao's lower bound for the variance of unbiased estimators of θ.

(ii) Is $\hat{\theta}=\frac{n+1}{n}Y_n$ an unbiased estimator of θ, where $Y_n=\max(X_1, X_2, ..., X_n)$.

(iii) Find the variance of $\hat{\theta}=\frac{n+1}{n}Y_n$.

[50 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from a population with pdf

$$f(x;\theta)=\frac{1}{(r-1)!\theta^r}x^{r-1}e^{-x/\theta},\ x\ge 0$$

for positive parameter θ and positive integer *r*. Find a sufficient statistic for θ.

[20 marks]

3. (a) *Suatu sampel rawak saiz 8 yang terdiri daripada* $x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 1,$ $x_5 = 0,\ x_6 = 1,\ x_7 = 1$ *dan* $x_8 = 0$ *diambil daripada populasi dengan fungsi jisim kebarangkalian (fjk)*

$$p_X(x;\theta) = \theta^x (1-\theta)^{1-x},\ x = 0, 1;\ 0 < \theta < 1.$$

Cari anggaran kebolehjadian maksimum untuk θ.

[30 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan seragam dengan fkk* $f_X(x;\theta) = \dfrac{1}{\theta},\ 0 \le x \le \theta.$

(i) *Cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama* θ.

(ii) *Adakah* $\hat{\theta} = \dfrac{n+1}{n} Y_n$ *suatu penganggar saksama* θ, *yang mana* $Y_n = maks(X_1, X_2, ..., X_n)$.

(iii) *Cari varians untuk* $\hat{\theta} = \dfrac{n+1}{n} Y_n$.

[50 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak saiz n daripada populasi dengan fkk*

$$f(x;\theta) = \frac{1}{(r-1)!\,\theta^r} x^{r-1} e^{-x/\theta},\quad x \ge 0$$

untuk parameter positif θ *dan integer positif r. Cari statistik cukup untuk* θ.

[20 markah]

4. (a) Let $\overline{X}$ be the mean of a random sample of size n from a $N(\mu, 9)$ distribution. Find n such that $P(\overline{X}-1<\mu<\overline{X}+1) = 0.90$, approximately.

[20 marks]

(b) Let $X_1, X_2, ..., X_n$ represent a random sample having pdf $f(x;\theta)=\theta e^{-\theta x} I_{(0,\infty)}(x)$, where $\Theta=\{\theta:\theta>0\}$. What is the generalized likelihood ratio test of size-α for testing $H_0:\theta\leq\theta_0$ versus $H_1:\theta>\theta_0$?

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ denote a random sample of size 5 from a common distribution with pdf

$$f(x)=\frac{1}{\alpha}e^{-x/\alpha}, \quad x > 0; \ \alpha > 0.$$

Find the most powerful test for testing $H_0:\alpha=1$ versus $H_1:\alpha=2$.

[40 marks]

4. (a) Biarkan $\overline{X}$ sebagai min suatu sampel rawak saiz n daripada taburan N(μ, 9). Cari n sedemikian sehingga $P\left(\overline{X}-1<\mu<\overline{X}+1\right)$ = 0.90, secara hampiran.

[20 markah]

(b) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak dengan fkk $f(x;\theta)=\theta e^{-\theta x} I_{(0,\infty)}(x)$, yang mana $\Theta=\{\theta : \theta>0\}$. Apakah ujian nisbah kebolehjadian teritlak saiz-α untuk menguji $H_0 : \theta \leq \theta_0$ lawan $H_1 : \theta > \theta_0$?

[40 markah]

(c) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz 5 daripada taburan sepunya dengan fkk

$$f(x)=\frac{1}{\alpha}e^{-x/\alpha}, \quad x > 0; \ \alpha > 0.$$

Cari ujian paling berkuasa untuk menguji $H_0 : \alpha = 1$ lawan $H_1 : \alpha = 2$.

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x} p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}\, x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}\, x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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