UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

April/May 2010

MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TWELVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi DUA BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Suppose that A, B and C are three events. Prove that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap \bar{C})P(\bar{C}|B).$$

(b) If the variance of the random variable X exist, show that

$$E(X^2) \geq [E(X)]^2.$$

[15 marks]

2. A bank administer two scales to each loan applicant to better predict the borrowers' success in paying back the loan. Let S_i be the event that the scale i shows that an applicant is qualified for the loan and let L be the event that an applicant is indeed qualified. Given

$$\alpha_i = P(S_i | L) \text{ , } \theta_i = P(\bar{S}_i | \bar{L}) \text{ and } p = P(L),$$

(a) what is the meaning of $P(L|\bar{S}_i)$?

(b) show that

$$P(L|\bar{S}_i) = \frac{(1-\alpha_i)p}{(1-\alpha_i)p + \theta_i(1-p)}.$$

(c) what condition need to be satisfied by $P(L|\bar{S}_1)$ and $P(L|\bar{S}_2)$ if the first scale is considered less likely to eliminate qualified applicants? Hence, show that

$$(1-\alpha_1)\theta_2 < (1-\alpha_2)\theta_1$$

[30 marks]

3. Let the distribution function of a random variable X be

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{8} & 0 < x < 2 \\ \frac{x^2}{16} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

(a) Find the density function of X.

(b) Compute $P(X \geq 1 | X \leq 3)$.

(c) Let $X_1, X_2, \cdots, X_9$ be nine independent observations for X. Find the probability that 3 out of these 9 observations have the values between 1 and 3.

[25 marks]

1. (a) Katakan A, B *dan* C *adalah tiga peristiwa. Buktikan bahawa*

$$P(A|B) = P(A|BC)P(C|B) + P(A|B\bar{C})P(\bar{C}|B).$$

(b) *Jika varians bagi suatu pemboleh ubah rawak X wujud, tunjukkan bahawa*

$$E(X^2) \geq [E(X)]^2.$$

[15 markah]

2. *Pihak bank menjalankan dua skala ke atas setiap pemohon pinjaman bagi meramalkan kejayaan peminjam membayar balik pinjaman. Katakan S_i adalah peristiwa bahawa skala i menunjukkan bahawa pemohon adalah layak untuk pinjaman dan L adalah peristiwa bahawa pemohon adalah sebenarnya layak. Diberi*

$$\alpha_i = P(S_i \mid L)\ ,\ \theta_i = P(\bar{S}_i \mid \bar{L}) \text{ dan } p = P(L),$$

(a) *apakah makna bagi $P(L \mid \bar{S}_i)$?*

(b) *tunjukkan bahawa*

$$P(L \mid \bar{S}_i) = \frac{(1-\alpha_i)p}{(1-\alpha_i)p + \theta_i(1-p)}.$$

(c) *Apakah syarat yang mesti dipenuhi oleh $P(L \mid \bar{S}_1)$ dan $P(L \mid \bar{S}_2)$ jika skala pertama dikatakan kurang berkesan untuk menghindar pemohon yang layak? Seterusnya, tunjukkan bahawa*

$$(1-\alpha_1)\theta_2 < (1-\alpha_2)\theta_1.$$

[30 markah]

3. *Katakan fungsi taburan bagi suatu pemboleh ubah rawak X adalah*

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{x}{8} & 0 < x < 2 \\ \dfrac{x^2}{16} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

(a) *Cari fungsi ketumpatan bagi X.*

(b) *Hitung $P(X \geq 1 \mid X \leq 3)$.*

(c) *Biar $X_1, X_2, \cdots, X_9$ sebagai sembilan cerapan tak bersandar bagi X. Cari kebarangkalian bahawa 3 dari 9 cerapan ini mempunyai nilai antara 1 dan 3.*

[25 markah]

4. A warehouse contains ten printing machines, four of which are defective. A company selects five of the machines at random, thinking all are in working condition.

(a) What is the probability that all five of the machines are nondefective?
(b) The company purchasing the machines returns the defective ones for repair. If it cost RM50 to repair each machine,
(i) find the mean and variance of the total repair cost.
(ii) in what interval would you expect the repair cost on these five machines to lie?

[25 marks]

5. The number of arrivals n at *Kedai Mahasiswa* checkout counter in the time interval from 0 to t follows a Poisson probability distribution with mean λt. Let Y denote the length of time until the first arrival.

(a) State the probability density function of Y.
(b) If students come into *Kedai Mahasiswa* at the rate of 10 per hour, what is the probability that more than 15 minutes will elapse between the next two students?
(c) What is the expected amount of time that the manager has to wait until he has 10 students coming in?
(d) If the students come into the store at the rate of 30 per hour, find the probability that it will take more than 15 minutes before 3 students come in.

[35 marks]

6. The joint density of random variables X and Y is given by

$$f(x,y)=\begin{cases} e^{-(x+y)} & ;0<x<\infty,\quad 0<y<\infty \\ 0 & ;\text{otherwise.} \end{cases}$$

(a) Obtain the distribution function of $Z=\frac{X}{Y}$
(b) Hence, find $P(X<Y)$.
(c) Compute $P(X<b)$.

[25 marks]

4. *Suatu gudang barangan mempunyai sepuluh mesin cetak, empat darinya adalah rosak. Suatu syarikat memilih lima mesin secara rawak, dengan anggapan semuanya berfungsi.*

(a) *Apakah kebarangkalian bahawa kesemua lima mesin ini adalah tidak rosak?*

(b) *Syarikat yang membeli mesin ini memulangkan mesin yang rosak untuk dibaiki. Jika kos pembaikan ialah RM50 bagi setiap mesin,*

(i) *cari min dan varians bagi jumlah kos pembaikan.*

(ii) *dalam selang apakah anda jangka kos pembaikan bagi lima mesin ini berada?*

[25 markah]

5. *Bilangan ketibaan n di kaunter bayaran Kedai Mahasiswa dalam selang masa dari 0 ke t mengikut suatu taburan kebarangkalian Poisson dengan min λt. Katakan Y adalah tempoh masa sehingga ketibaan pertama.*

(a) *Nyatakan fungsi ketumpatan kebarangkalian bagi Y.*

(b) *Jika pelajar masuk ke Kedai Mahasiswa pada kadar 10 sejam, apakah kebarangkalian bahawa lebih dari 15 minit akan berlalu antara dua pelajar berikutnya?*

(c) *Apakah amaun masa menunggu jangkaan bagi pengurus untuk menunggu sehingga dia mempunyai 10 pelajar yang masuk?*

(d) *Jika pelajar datang ke kedai ini pada kadar 30 sejam, cari kebarangkalian bahawa lebih dari 15 minit diperlukan sehingga 3 pelajar masuk ke kedai.*

[35 markah]

6. *Ketumpatan tercantum bagi pemboleh ubah rawak X dan Y diberi sebagai*

$$f(x,y) = \begin{cases} e^{-(x+y)} & ; 0 < x < \infty, \quad 0 < y < \infty \\ 0 & ; \text{sebaliknya.} \end{cases}$$

(a) *Dapatkan fungsi taburan bagi $Z = \dfrac{X}{Y}$.*

(b) *Seterusnya, cari $P(X < Y)$.*

(c) *Hitung $P(X < b)$.*

[25 markah]

7. The joint probability density function of random variables X and Y is given by

$$P(1,1)=\frac{1}{8} \quad P(1,2)=\frac{1}{4}$$
$$P(2,1)=\frac{1}{8} \quad P(2,2)=\frac{1}{2}$$

(a) Find the conditional probability density function of Y given $X=i$.
(b) Compute $P(XY\geq 3)$ and $P(X/Y\leq 1)$.
(c) Obtain $E\left(Y|x\right)$.

[30 marks]

8. (a) Let X_1, X_2 be a random sample from a distribution having the p.d.f.

$$f(x)=\begin{cases} e^{-x} & ; \quad 0<x<\infty \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Show that $Z=\frac{X_1}{X_2}$ has an F distribution.

(b) The coefficient of variation for a sample of values $Y_1, Y_2, \cdots, Y_n$ is defined by $CV=\frac{S}{\bar{Y}}$ where $\bar{Y}$ is the mean and S is the standard deviation. Let $Y_1, Y_2, \cdots, Y_{10}$ denote a random sample of size 10 from a normal distribution with mean 0 and variance σ^2.

(i) Obtain the distribution of $10\bar{Y}^2/S^2$.
(ii) What is the distribution of $S^2/10\bar{Y}^2$.
(iii) Find the number c such that

$$P\left(-c\leq\frac{S}{\bar{Y}}\leq c\right)=0.95.$$

[50 marks]

7. *Fungsi taburan kebarangkalian tercantum bagi pemboleh ubah rawak* X *dan* Y *diberi oleh*

$$P(1,1)=\frac{1}{8} \quad P(1,2)=\frac{1}{4}$$
$$P(2,1)=\frac{1}{8} \quad P(2,2)=\frac{1}{2}$$

(a) *Cari fungsi ketumpatan kebarangkalian bersyarat bagi* Y *diberi* $X=i$.

(b) *Hitung* $P(XY\geq 3)$ *dan* $P(X/Y\leq 1)$.

(c) *Dapatkan* $E\left(Y|x\right)$.

[30 markah]

8. (a) *Katakan* X_1 , X_2 *adalah suatu sampel rawak dari suatu taburan yang mempunyai f.k.k.*

$$f(x)=\begin{cases} e^{-x} & ; \quad 0<x<\infty \\ 0 & ; \quad \text{di tempat lain} \end{cases}$$

Tunjukkan bahawa $Z=\dfrac{X_1}{X_2}$ *mempunyai suatu taburan* F.

(b) *Koefisien ubahan bagi suatu sampel dengan nilai* $Y_1, Y_2, \cdots, Y_n$ *ditakrifkan oleh* $CV=\dfrac{S}{\bar{Y}}$ *yang mana* $\bar{Y}$ *ialah min dan* S *ialah sisihan piawai. Katakan* $Y_1, Y_2, \cdots, Y_{10}$ *adalah suatu sampel rawak bersaiz 10 dari suatu taburan normal dengan min 0 dan varians* σ^2.

(i) *Dapatkan taburan bagi* $10\bar{Y}^2/S^2$.

(ii) *Apakah taburan bagi* $S^2/10\bar{Y}^2$?

(iii) *Cari nombor* c *supaya*

$$P\left(-c\leq\frac{S}{\bar{Y}}\leq c\right)=0.95.$$

[50 markah]

9. (a) The relationship between a random variable Y and a random variable X is given by $Y = e^X$. If X has a normal distribution, find the probability density function of Y.

(b) Let X_1 and X_2 denote a random sample of size 2 from a normal distribution. If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + 2X_2$, show that the joint probability density function of Y_1 and Y_2 is bivariate normal.

[25 marks]

10. (a) Determine the density function of $W = \sqrt{X}$ if X is a standard normal random variable.

(b) Suppose X_1, X_2 is a random sample from a distribution $N(0,1)$.

(i) Find the joint probability distribution of $U = X_1^2 + X_2^2$ and $V = X_2$.

(ii) Obtain the marginal p.d.f. of U.

(iii) Are U and V independent?

[35 marks]

9. (a) *Hubungan antara suatu pemboleh ubah rawak* Y *dan suatu pemboleh ubah rawak* X *diberi oleh* $Y = e^X$. *Jika* X *mempunyai suatu taburan normal, cari fungsi ketumpatan kebarangkalian bagi* Y.

(b) *Katakan* X_1 *dan* X_2 *adalah suatu sampel rawak bersaiz 2 dari suatu taburan normal. Jika* $Y_1 = X_1 + X_2$ *dan* $Y_2 = X_1 + 2X_2$, *tunjukkan bahawa fungsi ketumpatan kebarangkalian tercantum bagi* Y_1 *and* Y_2 *adalah normal bivariat.*

[25 markah]

10. (a) *Tentukan fungsi ketumpatan bagi* $W = \sqrt{X}$ *jika* X *adalah suatu pemboleh ubah rawak normal piawai.*

(b) *Katkan* X_1, X_2 *adalah suatu sampel rawak dari suatu taburan* $N(0,1)$.

(i) *Cari taburan kebarangkalian tercantum bagi* $U = X_1^2 + X_2^2$ *dan* $V = X_2$.

(ii) *Dapatkan f.k.k. sut bagi* U.

(iii) *Adakah* U *dan* V *tak bersandar?*

[35 markah]

APPENDIX/LAMPIRAN

DISCRETE DISTRIBUTIONS	
Bernoulli	$f(x)=p^x(1-p)^{1-x}, \quad x=0,1$ $M(t)=1-p+pe^t$ $\mu=p, \quad \sigma^2=p(1-p)$
Binomial	$f(x)=\dfrac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \quad x=0,1,2,\ldots,n$ $M(t)=(1-p+pe^t)^n$ $\mu=p, \quad \sigma^2=np(1-p)$
Geometric	$f(x)=(1-p)^x p, \quad x=0,1,2,\ldots$ $M(t)=\dfrac{p}{1-(1-p)e^t}, t<\ln(1-p)$ $\mu=\dfrac{1-p}{p}, \quad \sigma^2=\mu=\dfrac{1-p}{p^2}$
Negative Binomial	$f(x)=\dfrac{(x+r-p)!}{x!(r-1)!}p^r(1-p)^x, \quad x=0,1,2,\ldots$ $M(t)=\dfrac{p^r}{\left[1-(1-p)e^t\right]^r}, t<-\ln(1-p)$ $\mu=\dfrac{r(1-p)}{p}, \quad \sigma^2=\dfrac{r(1-p)}{p^2}$
Poisson	$f(x)=\dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x=0,1,2,\ldots$ $M(t)=e^{\lambda(e^t-1)}$ $\mu=\lambda, \quad \sigma^2=\lambda$
Hipergeometric	$f(x)=\dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{t}}, \quad x\le r, x\le n_1, r-x\le n_2,$ $\mu=\dfrac{rn_1}{n}, \quad \sigma^2=\dfrac{rn_1n_2(n-r)}{n^2(n-1)}$

CONTINUOUS DISTRIBUTION	
Uniform	$f(x)=\frac{1}{b-a},\quad a\le x\le b$ $M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)},\quad t\ne 0,\quad M(0)=1$ $\mu=\frac{a+b}{2},\quad \sigma^2=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-x/\theta},\ 0\le x\le\infty$ $M(t)=\frac{1}{1-\theta t},\quad t<1/\theta$ $\mu=\theta,\quad \sigma^2=\theta^2$
Gamma	$f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-x/\theta},0\le x\le\infty$ $M(t)=\frac{1}{(1-\theta t)^{\alpha}},\quad t<1/0$ $\mu=\alpha\theta,\quad \sigma^2=\alpha\theta^2$
Chi Square	$f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2},\quad 0\le x\le\infty$ $M(t)=\frac{1}{(1-2t)^{r/2}},\quad t<\frac{1}{2}$ $\mu=r,\quad \sigma^2=2r$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left[(x-\mu)^2/2\sigma^2\right]},-\infty<x<\infty$ $M(t)=e^{\mu t+\sigma^2t^2/2}$ $E(X)=\mu,\quad Var(X)=\sigma^2$
Beta	$f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},0<x<1$ $\mu=\frac{\alpha}{\alpha+\beta},\quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

	FORMULA
1.	$\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$
2.	$\sum_{n=0}^{\infty}ar^n=\frac{a}{1-r},\quad \lvert r\rvert<1$
3.	$\sum_{x=0}^{n}\binom{n}{x}b^x a^{n-x}=\ a+b\ ^{n}$
4.	$\sum_{x=0}^{n}\binom{n}{x}\binom{r-n}{r-x}=\binom{n}{r}$
5.	$\sum_{x=0}^{n}\binom{n+k-1}{k}w^k=\ 1-w\ ^{-n},\quad \lvert w\rvert<1$
6.	$\Gamma\ \alpha\ =\int_0^{\infty}x^{\alpha-1}e^{-x}dx,\quad \Gamma\ \alpha\ =\ \alpha-1\ !$
7.	$B\ \alpha,\beta\ =\int_0^{1}x^{\alpha-1}\ 1-x\ ^{\beta-1}dx$
8.	$B\ \alpha,\beta\ =\frac{\Gamma\ \alpha\ \Gamma\ \beta}{\Gamma\ \alpha+\beta}$
9.	Polar coordinates: $y=r\cos\theta$ $z=r\sin\theta$

- ooo O ooo -