UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

April/May 2010

MAT 122 – Differential Equations 1
[Persamaan Pembezaan I]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) (i) Find the particular solution for

$$1+x^3 \ dy - x^2 y dx = 0$$

which satisfies the initial condition $y \ 1 = 2.$

(ii) Is the solution unique? Justify your answer using the relevant theorem.

(b) Consider the differential equation

$$y' = y^2.$$

(i) Determine the region in the xy plane where the equation has a unique solution through the point (x_0, y_0) located in the interior of the region. Justify your answer using the relevant theorem.

(ii) Solve the initial value problem

$$y' = y^2$$
$$y(0) = 1.$$

(iii) Explain why $y = \dfrac{1}{1-x}$ is not a solution for the above initial value problem on the interval $(-2, 2)$ but is a solution on the interval $(-1, 1)$.

(c) Determine a function $N(x, y)$ such that the differential equation below is exact:

$$\left(y^{1/2}x^{-1/2} + \frac{x}{x^2+y}\right)dx + N(x, y)dy = 0$$

[100 marks]

1. (a) (i) *Dapatkan penyelesaian khusus bagi*

$$1+x^3 \;\; dy - x^2 y dx = 0$$

yang memenuhi syarat awal $y \; 1 = 2.$

(ii) *Adakah penyelesaian bagi (i) unik? Berikan penjelasan anda dengan disokong oleh teorem yang berkaitan.*

(b) *Pertimbangkan persamaan pembezaan*

$$y' = y^2.$$

(i) *Tentukan suatu rantau pada satah* xy *di mana persamaan di atas mempunyai penyelesaian unik melalui titik* (x_0, y_0) *yang terletak dalam rantau tersebut. Sokong jawapan anda dengan teorem yang sesuai.*

(ii) *Selesaikan masalah nilai awal*

$$y' = y^2$$
$$y(0) = 1.$$

(iii) *Terangkan mengapa* $y = \dfrac{1}{1-x}$ *bukan suatu penyelesaian bagi masalah nilai awal di atas pada selang* $(-2, 2)$ *tetapi merupakan suatu penyelesaian pada selang* $(-1, 1)$.

(c) *Tentukan suatu fungsi* $N(x, y)$ *supaya persamaan pembezaan berikut adalah tepat:*

$$\left(y^{1/2}x^{-1/2} + \frac{x}{x^2+y}\right)dx + N(x, y)dy = 0$$

[100 markah]

2. (a) Show that the general solution for $y'' - 3y' + 2y = 0$ is

$$y = c_1 e^{x} + c_2 e^{2x}.$$

Hence, find the general solution for

$$y'' - 3y' + 2y = \frac{e^{2x}}{e^{x} + 1}.$$

(b) Solve $y'' + y' - 12y = \sinh 4x$ given the identity $\sinh 4x = \frac{(e^{4x} - e^{-4x})}{2}$.

(c) (i) Show that the equation

$$F(x^2 y^{1/2})y\,dx + G(x^2 y^{1/2})x\,\mathrm{dy} = 0$$

can be transformed into a variable separable equation in the form $H(x)dx + I(v)\,\mathrm{d}v = 0$ using the transformation $v = x^2 y^{1/2}$.

(ii) Hence, solve

$$(2 + 2x^2 y^{1/2})y\,dx + (x^2 y^{1/2} + 2)x\,\mathrm{dy} = 0$$

[100 marks]

3. (a) Consider the initial value problem

$$y' = x^2 + y^2, \quad y(x_0) = y_0.$$

(i) By using Euler's formula, show that

$$y_k = y_{k-1} + h(x_{k-1}^2 + y_{k-1}^2), \quad k = 1,2,3,\cdots.$$

(ii) Show that

$$y_n = y_0 + h\sum_{i=0}^{n-1}(x_i^2 + y_i^2), \quad n = 1,2,3,\cdots.$$

(iii) Given $x_0 = 0$ and $y_0 = 1$, estimate $y(0.3)$ using $h = 0.1$.

(iv) What is the upper bound for the accumulated formula error for your answer in (iii)?

(b) Solve the Hermite equation

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty$$

where λ is a constant, in powers of x.

[100 marks]

2. (a) *Tunjukkan bahawa penyelesaian am bagi* $y'' - 3y' + 2y = 0$ *ialah*

$$y = c_1 e^{x} + c_2 e^{2x}.$$

Seterusnya, dapatkan penyelesaian am bagi

$$y'' - 3y' + 2y = \frac{e^{2x}}{e^{x} + 1}.$$

(b) *Selesaikan* $y'' + y' - 12y = \sinh 4x$ *jika diberikan identiti* $\sinh 4x = \dfrac{(e^{4x} - e^{-4x})}{2}$.

(c) (i) *Tunjukkan bahawa persamaan*

$$F(x^2 y^{1/2})y\,dx + G(x^2 y^{1/2})x\,\mathrm{d}y = 0$$

boleh ditukarkan kepada persamaan pembolehubah terpisah $H(x)dx + I(v)\,\mathrm{d}v = 0$ *dengan menggunakan transformasi* $v = x^2 y^{1/2}$.

(ii) *Seterusnya, selesaikan*

$$(2 + 2x^2 y^{1/2})y\,dx + (x^2 y^{1/2} + 2)x\,\mathrm{d}y = 0$$

[100 markah]

3. (a) *Pertimbangkan masalah nilai awal*

$$y' = x^2 + y^2, \quad y(x_0) = y_0.$$

(i) *Tunjukkan dengan menggunakan rumus Euler bahawa*

$$y_k = y_{k-1} + h(x_{k-1}^2 + y_{k-1}^2), \quad k = 1,2,3,\cdots.$$

(ii) *Tunjukkan bahawa*

$$y_n = y_0 + h\sum_{i=0}^{n-1}(x_i^2 + y_i^2), \quad n = 1,2,3,\cdots.$$

(iii) *Diberi* $x_0 = 0$ *dan* $y_0 = 1$, *anggarkan* $y(0.3)$ *dengan menggunakan* $h = 0.1$.

(iv) *Apakah suatu batas atas bagi ralat rumus tertumpuk bagi jawapan anda di (iii)?*

(b) *Selesaikan persamaan Hermite*

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty$$

di mana λ *ialah suatu pemalar, dalam kuasa* x.

[100 markah]

4. (a) (i) Show that $e^{\lambda(t-t_0)}\mathbf{v}$, t_0 constant, is a solution for

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \text{ if } \mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

(ii) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}.$$

(b) Two spaces A_1 and A_2, with volumes 10 litres each, are segregated with a membrane. Let $y_1(t), y_2(t)$ be the amount of nutrient solution in A_1 and A_2 at time t respectively. Then,

$$P_1(t) = \frac{y_1(t)}{10}, \quad P_2(t) = \frac{y_2(t)}{10}$$

represent the concentrations of the solution in A_1 and A_2 respectively. The nutrient solution diffuses through the membrane from one space to another where the rate of diffusion is proportional to the difference in the concentration of the solution in both spaces, that is,

$$y_1'(t) = k(P_2(t) - P_1(t))$$
$$y_2'(t) = k(P_1(t) - P_2(t))$$

where k is the proportional constant (k>0).

(i) Show that

$$P_1''(t) + \frac{k}{5}P_1'(t) = 0$$

and

$$P_1'(t) = P_1(0) - \frac{1}{2}[P_2(0) - P_1(0)](e^{-kt/5} - 1).$$

(ii) The equilibrium situation is obtained in an infinite time. What is the concentration $P_1(t)$ at that time?

[100 marks]

4. (a) (i) *Tunjukkan bahawa* $e^{\lambda(t-t_0)}\mathbf{v}$, t_0 *pemalar, adalah suatu penyelesaian bagi*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \text{ jika } \mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

(ii) *Selesaikan masalah nilai awal*

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}.$$

(b) *Dua ruang* A_1 *dan* A_2, *masing-masing dengan isipadu 10 liter, diasingkan dengan suatu selaput. Katakan* $y_1(t), y_2(t)$ *ialah kuantiti larutan zat masing-masing di dalam* A_1 *dan* A_2 *pada masa t. Maka,*

$$P_1(t) = \frac{y_1(t)}{10}, \quad P_2(t) = \frac{y_2(t)}{10}$$

merupakan kepekatan larutan di dalam A_1 *dan* A_2. *Larutan zat dapat meresap melalui selaput dari suatu ruang ke ruang yang satu lagi. Kadar peresapan berkadaran dengan perbezaan di antara kepekatan larutan di dalam kedua-dua ruang, iaitu*

$$y_1'(t) = k(P_2(t) - P_1(t))$$
$$y_2'(t) = k(P_1(t) - P_2(t))$$

dengan k sebagai pemalar berkadaran (k>0).

(i) *Tunjukkan bahawa*

$$P_1''(t) + \frac{k}{5}P_1'(t) = 0$$

dan

$$P_1'(t) = P_1(0) - \frac{1}{2}[P_2(0) - P_1(0)](e^{-kt/5} - 1).$$

(ii) *Keadaan keseimbangan dicapai selepas suatu masa yang terlalu panjang. Apakah nilai kepekatan* $P_1(t)$ *ketika itu?*

[100 markah]

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