UNIVERSITI SAINS MALAYSIA

Second Semester Examination
2009/2010 Academic Session

April/May 2010

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) (i) If $\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}$, find a, b, c and d.

(ii) Let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix defined by $a_{ii}=k$ and $a_{ij}=0$ for $i\neq j$. Show that if B is an $n\times n$ matrix, then $AB=kB$.
[Hint: Let $B=\left[b_{ij}\right]$ and $C=\left[c_{ij}\right]$ such that $C=AB$]

(b) Given the following matrices

$$A=\begin{bmatrix} 0 & 1 \\ -2 & 1 \\ -1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \text{ and } C=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

(i) If possible, evaluate $2AB - C$.

(ii) Find C^{-1} using the Gauss-Jordan procedure.

(iii) If possible, evaluate $(I_3 - 2ABC^{-1})C$.

(iv) Is BA singular or non-singular? Justify.

(c) Prove the following where $A\in \mathrm{M}_{n\times n}$ □ :

(i) Consider the homogeneous system $AX=\underline{0}$. If A is non-singular, then the only solution is the trivial one, $X=\underline{0}$.

(ii) If A is symmetric and non-singular, then A^{-1} is symmetric.
[Hint: The identity matrix I_n is symmetric]

(d) Determine all k for which the following set is linearly independent.
$S =\ 1, -1, 3, 1\ ,\ \ -1, 1, 2, 1\ ,\ \ -1, 1, k, 5\ \ .$

(e) For what set of numbers a, b, c will the system

$$\begin{array}{ccccccc} 2x_1 & - & 3x_2 & + & 3x_3 & = & a \\ 4x_1 & + & x_2 & + & 6x_3 & = & b \\ 2x_1 & + & 7x_2 & + & 3x_3 & = & c \end{array}$$

have a solution?
[Hint: You do not need to follow the Gaussian Elimination/Gauss-Jordan procedure]

[100 marks]

1. (a) (i) *Jika* $\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}$, *cari a, b, c dan d.*

(ii) *Biar* $A = \left[a_{ij}\right]$ *suatu matriks* $n \times n$ *yang ditakrifkan dengan* $a_{ii} = k$ *dan* $a_{ij} = 0$ *untuk* $i \neq j$. *Tunjukkan bahawa jika B suatu matriks* $n \times n$, *maka* $AB = kB$.
[Petunjuk: Biar $B = \left[b_{ij}\right]$ *dan* $C = \left[c_{ij}\right]$ *sedemikian hingga* $C = AB$*]*

(b) *Diberi matriks berikut*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \text{ dan } C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

(i) *Jika mungkin, nilaikan 2AB – C.*

(ii) *Cari* C^{-1} *menggunakan prosedur Gauss-Jordan.*

(iii) *Jika mungkin, nilaikan* $(I_3 - 2ABC^{-1})C$.

(iv) *Adakah BA singular atau tak singular? Beri justifikasi.*

(c) *Buktikan yang berikut yang mana* $A \in \mathrm{M}_{n\times n}$ □ :

(i) *Pertimbangkan sistem homogen* $AX = \underline{0}$. *Jika A tak singular, maka penyelesaian adalah hanya yang remeh,* $X = \underline{0}$.

(ii) *Jika A adalah simetri dan tak singular, maka* A^{-1} *adalah simetri.*
[Petunjuk: Matriks identiti I_n *adalah simetri]*

(d) *Tentukan semua k supaya set yang berikut adalah tak bersandar linear.*

$$S = \{(1, -1, 3, 1), (-1, 1, 2, 1), (-1, 1, k, 5)\}.$$

(e) *Apakah set nombor* (a, b, c) *supaya sistem*

$$\begin{aligned} 2x_1 - 3x_2 + 3x_3 &= a \\ 4x_1 + x_2 + 6x_3 &= b \\ 2x_1 + 7x_2 + 3x_3 &= c \end{aligned}$$

mempunyai suatu penyelesaian?
[Petunjuk: Anda tidak perlu mengikut prosedur Penghapusan Gauss/Gauss-Jordan]

[100 markah]

2. (a) (i) Find a basis for $\mathbb{R}^3$ containing the vector $(1,-2,1)$.

(ii) Given that the set $S=\left\{(1,2,3),\ (0,1,2),\ \left(-1,\frac{1}{2},3\right),\ (1,1,1)\right\}$ generates $\mathbb{R}^3$. Find a subset T of S that forms a basis for $\mathbb{R}^3$.

(b) Define U as the set of all polynomials of degree 3 with the usual addition and scalar multiplication operations as in $\mathrm{P}_3(\mathbb{R})$. Is U a vector space? If it is, prove it. If not, explain why.

(c) For each of the following, determine whether or not S is a subspace of the indicated vector space. Prove your claim.

(i) $S=\{ax^3+bx^2+cx+d \mid ad=0\}$ in $\mathrm{P}_3(\mathbb{R})$.

(ii) $S=\{(x_1,x_2,x_3,x_4) \mid x_1=2x_2+3x_4 \text{ and } x_3=2x_4\}$ in $\mathbb{R}^4$.

(iii) $S=\left\{\begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} \middle| \ p,q \text{ are integers}\right\}$ in $\mathrm{M}_{2\times 2}(\mathbb{R})$.

(d) Let V be a vector space. Suppose $T=\{v_1,v_2,\cdots,v_n\}$ is a set of non-zero vectors in V such that every vector in V can be written in one and only one way as a linear combination of the vectors in T. Prove that T is a basis of V.

(e) Given

$$A=\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \text{ where } \mathrm{RREF}(A)=\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Determine the column space of A, $C(A)$ and its column rank, $c(A)$.

(ii) Determine the row space of A, $R(A)$ and its row rank, $r(A)$.

(iii) Given that the null space of A, $N(A)$ is the solution to the homogeneous linear system represented in the form $Av=\underline{0}$. Let

$$v=\begin{bmatrix} x \\ y \\ z \\ w \\ t \end{bmatrix}\in\mathbb{R}^5.$$ Find $N(A)$ and its dimension, $nullity(A)$.

(iv) Deduce the rank of A, $\rho(A)$ an verify that $\rho(A)+nullity(A)=5$.

[100 marks]

2. (a) (i) *Dapatkan suatu asas untuk* $\mathbb{R}^3$ *yang mengandungi vektor* $(1,-2,1)$.

(ii) *Diberi bahawa set* $S=\left\{(1,2,3),\ (0,1,2),\ \left(-1,\frac{1}{2},3\right),\ (1,1,1)\right\}$ *menjana* $\mathbb{R}^3$. *Cari suatu subset T dari S yang membentuk asas bagi* $\mathbb{R}^3$.

(b) *Takrifkan U sebagai set semua polinomial berdarjah 3 dengan operasi penambahan dan pendaraban skalar yang lazim dalam* $\mathrm{P}_3(\mathbb{R})$. *Adakah U suatu ruang vektor? Jika ya, buktikan. Jika tidak, terangkan mengapa.*

(c) *Untuk setiap daripada berikut, tentukan sama ada S adalah subruang bagi ruang vektor yang tertunjuk. Buktikan tuntutan anda.*

(i) $S=\{ax^3+bx^2+cx+d \mid ad=0\}$ *dalam* $\mathrm{P}_3(\mathbb{R})$.

(ii) $S=\{(x_1,x_2,x_3,x_4) \mid x_1=2x_2+3x_4 \text{ dan } x_3=2x_4\}$ *dalam* $\mathbb{R}^4$.

(iii) $S=\left\{\begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} \middle| \ p,q \text{ adalah integer}\right\}$ *dalam* $\mathrm{M}_{2\times 2}(\mathbb{R})$.

(d) *Biar V suatu ruang vektor. Andaikan* $T=\{v_1,v_2,\cdots,v_n\}$ *adalah set vektor-vektor bukan sifar dalam V sedemikian hingga semua vektor dalam V boleh ditulis sebagai satu gabungan linear vektor-vektor dalam T.*
Tunjukkan bahawa T ialah suatu asas bagi V.

(e) *Diberi*

$$A=\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \text{ dengan } \mathrm{BEBT}(A)=\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) *Tentukan ruang lajur bagi A, C(A) dan pangkat lajurnya, c(A).*

(ii) *Tentukan ruang baris bagi A, R(A) dan pangkat barisnya, r(A).*

(iii) *Diberi bahawa ruang nol bagi A, N(A) adalah penyelesaian bagi sistem homogen dalam bentuk* $Av=\underline{0}$. *Biar* $v=\begin{bmatrix} x \\ y \\ z \\ w \\ t \end{bmatrix}\in\mathbb{R}^5$.

Cari N(A) dan dimensinya, kenolan(A).

(iv) *Buat kesimpulan tentang pangkat A,* $\rho(A)$ *dan tentusahkan yang*

$$\rho(A)+kenolan(A)=5.$$

[100 markah]

3. (a) Find an example for each of the following or explain why no such example can be found.

(i) A basis for $P_3(\mathbb{R})$ containing three vectors.

(ii) A set of 4 vectors in $M_{2\times 2}(\mathbb{R})$ that spans $M_{2\times 2}(\mathbb{R})$.

(iii) A set of 4 vectors in $\mathbb{R}^5$ that is linearly independent.

(iv) A set of 4 vectors in $\mathbb{R}^4$ that is not a basis for $\mathbb{R}^4$.

(b) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation where A is the standard matrix for T. If T^2 is defined by $(x,y)T^2 = ((x,y)T)T$ for all $(x,y) \in \mathbb{R}^2$, show that T^2 is a linear transformation and its standard matrix is A^2.

(c) Given the linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$(x,y,z,w)T = (x-2y, 3z+w)$$

(i) Find the standard matrix for T.

(ii) Find the kernel of T, Ker T.

(iii) Find a basis for Ker T.

(iv) Find a basis for the image of T, Im T.

(v) Is T one-to-one? Explain your answer.

(vi) Is T onto? Explain your answer.

(d) Given $W = \mathscr{L}\{(1,1,1,1), (-2,0,1,3), (0,0,1,0)\}$.

(i) Using the Gram-Schmidt process, find an orthonormal basis of W.

(ii) Find the orthogonal complement of W, $W^\perp$.

(iii) Explain why $\mathbb{R}^4 = W \oplus W^\perp$.

(e) Let V be an inner product space. Show that:

(i) If W is a subspace of V, then the zero vector of V belongs to $W^\perp$.

(ii) $V^\perp = \{\underline{0}\}$ and $\{\underline{0}\}^\perp = V$.

(iii) If W is a subspace of V where $W = \mathscr{L}(S)$, then a vector **u** in V belongs to $W^\perp$ if and only if **u** is orthogonal to every vector in S.

[100 marks]

3. (a) *Cari satu contoh untuk setiap yang berikut atau terangkan mengapa tiada contoh yang boleh dicari.*

   (i) *Suatu asas bagi* $P_3(\mathbb{R})$ *yang mengandungi tiga vektor.*

   (ii) *Suatu set 4 vektor dalam* $M_{2\times 2}(\mathbb{R})$ *yang merentang* $M_{2\times 2}(\mathbb{R})$.

   (iii) *Suatu set 4 vektor dalam* $\mathbb{R}^5$ *yang tak bersandar linear.*

   (iv) *Suatu set 4 vektor dalam* $\mathbb{R}^4$ *yang bukan asas bagi* $\mathbb{R}^4$.

(b) *Biar* $T:\mathbb{R}^2 \to \mathbb{R}^2$ *suatu transformasi linear yang mana A ialah matriks piawai bagi T. Jika* T^2 *ditakrif dengan* $(x,y)T^2 = ((x,y)T)T$ *untuk semua* $(x,y) \in \mathbb{R}^2$, *tunjukkan bahawa* T^2 *ialah suatu transformasi linear dan matriks piawainya ialah* A^2.

(c) *Diberi transformasi linear* $T:\mathbb{R}^4 \to \mathbb{R}^2$ *yang ditakrifkan dengan*

$$(x,y,z,w)T = (x-2y, 3z+w)$$

   (i) *Cari matriks piawai bagi T.*

   (ii) *Cari inti bagi T, Ker T.*

   (iii) *Cari asas bagi Ker T.*

   (iv) *Cari asas untuk imej bagi T, Im T.*

   (v) *Adakah T satu-ke-satu? Terangkan jawapan anda.*

   (vi) *Adakah T keseluruhan? Terangkan jawapan anda.*

(d) *Diberi* $W = \mathscr{L}\{(1,1,1,1), (-2,0,1,3), (0,0,1,0)\}$.

   (i) *Menggunakan proses Gram-Schmidt, cari asas ortonormal bagi W.*

   (ii) *Cari pelengkap berortogon bagi W,* $W^{\perp}$.

   (iii) *Terangkan mengapa* $\mathbb{R}^4 = W \oplus W^{\perp}$.

(e) *Biar V suatu ruang hasildarab terkedalam. Tunjukkan bahawa:*

   (i) *Jika W ialah suatu subruang dari V, maka vektor sifar bagi V berada dalam* $W^{\perp}$.

   (ii) $V^{\perp} = \{\underline{0}\}$ *dan* $\{\underline{0}\}^{\perp} = V$.

   (iii) *Jika W ialah suatu subruang dari V yang mana W=*$\mathscr{L}$*(S), maka suatu vektor* ***u*** *daripada V berada dalam* $W^{\perp}$ *jika hanya jika* ***u*** *berortogon dengan semua vektor dalam S.*

[100 markah]

4. (a) Find the least squares fit line for the points $(-2,1)$, $(-1,3)$, $(0,2)$, $(1,3)$ and $(2,1)$.

(b) Given that $T:\mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation where α and β are bases for $\mathbb{R}^3$ such that $\beta = \{(1,2),(1,-1)\}$. Suppose

$$[(x,y,z)]_\alpha = (y-z, z, x+y-z) \quad \text{for all} \quad (x,y,z) \in \mathbb{R}^3 \text{ and } T_{\alpha,\beta} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

find the definition $(x,y,z)T$ for T.

(c) Given

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Find matrices C and D such that $C^{-1}AC = D$ where C is the diagonalizing matrix for A.

(d) Use the Cayley-Hamilton theorem to find B^{-1} if given $B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ and

$$\det(B - xI) = -x^3 + 3x^2 + x - 3.$$

$$\left[\text{Hint: Given that } B^2 = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix}\right]$$

(e) (i) Let λ be an eigenvalue of a linear transformation $T:V \to V$. Prove that the eigenvector of T corresponding to λ is the non-zero vector in the kernel of $\lambda I - T$.

(ii) Show that if A and B are two $n \times n$ matrices having the same diagonalizing matrix C, then $AB = BA$.
[Hint: The product of two diagonal matrices is commutative]

[100 marks]

4\. (a) *Cari garislurus padanan terbaik untuk titik-titik* $(-2,1)$, $(-1,3)$, $(0,2)$, $(1,3)$ *dan* $(2,1)$.

(b) *Diberi* $T:\mathbb{R}^3 \rightarrow \mathbb{R}^2$ *ialah suatu transformasi linear yang mana* α *dan* β *adalah asas-asas bagi* $\mathbb{R}^3$ *sedemikian hingga* $\beta = \{(1,2),(1,-1)\}$. *Andai* $[(x,y,z)]_\alpha = (y-z, z, x+y-z)$ *untuk semua* $(x,y,z) \in \mathbb{R}^3$ *dan* $T_{\alpha,\beta} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$, *cari takrif* $(x,y,z)T$ *bagi T.*

(c) *Diberi*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Cari matriks-matriks C dan D sedemikian hingga $C^{-1}AC = D$ *yang mana C ialah matriks mempepenjuru bagi A.*

(d) *Guna teorem Cayley-Hamilton untuk mencari* B^{-1} *jika* $B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

dan $\det(B - xI) = -x^3 + 3x^2 + x - 3.$

$$\left[\textit{Petunjuk : Diberi } B^2 = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} \right]$$

(e) (i) *Biar* λ *suatu nilai eigen bagi suatu transformasi linear* $T:V \rightarrow V$. *Buktikan yang vektor eigen bagi T yang bersepadan dengan* λ *ialah vektor bukan sifar dalam inti bagi* $\lambda I - T$.

(ii) *Tunjukkan bahawa jika A dan B adalah dua matriks* $n \times n$ *yang mempunyai matriks mempepenjuru C yang sama, maka* $AB = BA$.
[Petunjuk: Hasildarab dua matriks pepenjuru adalah saling bertukartertib]

[100 markah]

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