UNIVERSITI SAINS MALAYSIA

First Semester Examination
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MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) A dice is rolled twice. If "1", "2", "3" and "4" do not appear in any of the throws, such an event is classified as a success.
   (i) Find the set of all possible outcomes of successes, Ω.
   (ii) Find the σ-field, S, of this experiment based on (i).

[30 marks]

(b) Assume that $\Pr(C_1 \cap C_2 \cap C_3) > 0$. Show that $\Pr(C_1 \cap C_2 \cap C_3 \cap C_4) = \Pr(C_1).\Pr(C_2 | C_1).\Pr(C_3 | C_1 \cap C_2).\Pr(C_4 | C_1 \cap C_2 \cap C_3)$.

[20 marks]

(c) Let $\psi(t) = \log M(t)$, where $M(t)$ is the moment generating function (mgf) of a distribution. Prove that $\psi'(0) = \mu$ and $\Psi''(0) = \sigma^2$.

[30 marks]

(d) Let X be a random variable with probability density function (pdf)

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } 0 \leq x \leq 1 \\ \frac{1}{2x^2}, & \text{if } x > 1 \end{cases}$$

Find the pdf of $Y = \frac{1}{X}$ using the distribution function method.

[20 marks]

2. (a) Let X_1 and X_2 be independent random variables having a common pdf

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

If $U = X_1 + X_2$ and $V = X_1 - X_2$, find
   (i) the pdf of $X_1 + X_2$.
   (ii) the pdf of $X_1 - X_2$.
   (iii) the conditional pdf of V, given $U = u$, for a fixed value of $u > 0$.

[40 marks]

(b) If $E[h(X)]$ exists, i.e., $E|h(X)| < \infty$, show that $E[E\{h(X)|Y\}] = E[h(X)]$, where (X, Y) are discrete random variables.

[20 marks]

1. (a) *Suatu dadu dilambungkan dua kali. Jika "1", "2", "3" dan "4" tidak muncul dalam sebarang lambungan, peristiwa sedemikian diklasifikasikan sebagai kejayaan.*
   (i) *Cari set semua peristiwa kejayaan yang mungkin, Ω.*
   (ii) *Cari medan-σ, S, untuk eksperimen ini berdasarkan (i).*

   [30 markah]

   (b) *Andaikan* $P(C_1 \cap C_2 \cap C_3) > 0$. *Tunjukkan bahawa* $P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1).P(C_2|C_1).P(C_3|C_1 \cap C_2).P(C_4|C_1 \cap C_2 \cap C_3)$.

   [20 markah]

   (c) *Biarkan $\psi(t) = \log M(t)$, yang mana M(t) ialah fungsi penjana momen (fpm) suatu taburan. Buktikan bahawa $\psi'(0) = \mu$ dan $\Psi''(0) = \sigma^2$.*

   [30 markah]

   (d) *Biarkan X sebagai pembolehubah rawak dengan fungsi ketumpatan kebarangkalian (fkk)*

$$f(x) = \begin{cases} 0, & jika\ x < 0 \\ \frac{1}{2}, & jika\ 0 \leq x \leq 1 \\ \frac{1}{2x^2}, & jika\ x > 1 \end{cases}$$

   Cari fkk untuk $Y = \frac{1}{X}$ dengan menggunakan kaedah fungsi taburan.

   [20 markah]

2. (a) *Biarkan X_1 dan X_2 sebagai pembolehubah-pembolehubah rawak tak bersandar yang mempunyai fkk sepunya*

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

   Jika $U = X_1 + X_2$ dan $V = X_1 - X_2$, cari
   (i) *fkk untuk* $X_1 + X_2$.
   (ii) *fkk untuk* $X_1 - X_2$.
   (iii) *fkk bersyarat V, diberi U = u, untuk suatu nilai u > 0.*

   [40 markah]

   (b) *Jika E[h(X)] wujud, yakni E|h(X)| < ∞, tunjukkan bahawa E[E{h(X)|Y}] = E[h(X)], yang mana (X, Y) adalah pembolehubah-pembolehubah rawak diskrit.*

   [20 markah]

(c) Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be a set of order statistics corresponding to independent random variables $X_1, X_2, ..., X_n$, which have a common pdf

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0 & , \text{elsewhere} \end{cases}.$$

(i) Find the joint pdf of $X_{(r)}$ and $X_{(s)} - X_{(r)}$, where $s > r$.

(ii) Are $X_{(r)}$ and $X_{(s)} - X_{(r)}$ in (i) independent?

(iii) Find the pdf of $X_{(r+1)} - X_{(r)}$.

(Hint : $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$)

[40 marks]

3. (a) Assume that $X_1, X_2, ..., X_{m+1}$ represent a random sample of size $m+1$ from a $N(\mu_X, \sigma^2)$ distribution and $Y_1, Y_2, ..., Y_{n+1}$ represent a random sample of size $n+1$ from a $N(\mu_Y, \sigma^2)$ distribution. If the two samples are independent, find the distribution of each of the following statistics:

(i) $$\frac{1}{\sigma^2}\left[\sum_{i=1}^{m+1}(X_i - \mu_X)^2 + \sum_{i=1}^{n+1}(Y_i - \mu_Y)^2\right]$$

(ii) $$\frac{(n+1)\sum_{i=1}^{m+1}(X_i - \mu_X)^2}{(m+1)\sum_{i=1}^{n+1}(Y_i - \mu_Y)^2}$$

(iii) $$\frac{n\sum_{i=1}^{m+1}(X_i - \bar{X})^2}{m\sum_{i=1}^{n+1}(Y_i - \bar{Y})^2}$$

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ represent a random sample of size n from a standard normal distribution. Find the limiting distribution of

$$W_n = \frac{\sqrt{n}\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2}.$$

[20 marks]

(c) *Biarkan* $X_{(1)}, X_{(2)}, ..., X_{(n)}$ *sebagai suatu set statistik tertib yang sepadan dengan pembolehubah-pembolehubah rawak tak bersandar* $X_1, X_2, ..., X_n$ *yang mempunyai fkk sepunya*

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0 & , \text{di tempat lain} \end{cases}.$$

(i) *Cari fkk tercantum* $X_{(r)}$ dan $X_{(s)} - X_{(r)}$, yang mana $s > r$.

(ii) *Adakah* $X_{(r)}$ dan $X_{(s)} - X_{(r)}$ *dalam (i) tak bersandar?*

(iii) *Cari fkk* $X_{(r+1)} - X_{(r)}$.

(Petunjuk : $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$*)*

[40 markah]

3. (a) *Andaikan* $X_1, X_2, ..., X_{m+1}$ *mewakili suatu sampel rawak saiz m+1 daripada taburan* $N(\mu_X, \sigma^2)$ *dan* $Y_1, Y_2, ..., Y_{n+1}$ *mewakili suatu sampel rawak saiz n+1 daripada taburan* $N(\mu_Y, \sigma^2)$. *Jika kedua-dua sampel adalah tak bersandar, cari taburan untuk setiap statistik berikut:*

(i) $$\frac{1}{\sigma^2}\left[\sum_{i=1}^{m+1}(X_i - \mu_X)^2 + \sum_{i=1}^{n+1}(Y_i - \mu_Y)^2\right]$$

(ii) $$\frac{(n+1)\sum_{i=1}^{m+1}(X_i - \mu_X)^2}{(m+1)\sum_{i=1}^{n+1}(Y_i - \mu_Y)^2}$$

(iii) $$\frac{n\sum_{i=1}^{m+1}(X_i - \bar{X})^2}{m\sum_{i=1}^{n+1}(Y_i - \bar{Y})^2}$$

[30 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak saiz n daripada taburan normal piawai. Cari taburan penghad untuk*

$$W_n = \frac{\sqrt{n}\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2}.$$

[20 markah]

(c) Let X has a Poisson distribution with parameter θ. The prior probabilities on Θ are $\Pr(\Theta = 0.4) = \frac{2}{5}$ and $\Pr(\Theta = 0.3) = \frac{3}{5}$. If $X = 4$, find the following:

(i) Posterior probability for Θ = 0.4.

(ii) Posterior probability for Θ = 0.3.

[30 marks]

(d) For the pdf $f_\theta(x) = \theta(1-\theta)^x$, $x = 0, 1, 2, \ldots$, $0<\theta<1$, find the Cramer-Rao's lower bound for the variance of unbiased estimators of θ based on a random sample of size n.

[20 marks]

4. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f_\theta(x) = \begin{cases} \theta\alpha x^{\alpha-1}\exp(-\theta x^\alpha), & x>0,\ \alpha \text{ is known} \\ 0, & \text{elsewhere} \end{cases}.$$

Find the maximum likelihood estimator of θ.

[20 marks]

(b) Let $X_1, X_2, \ldots, X_n$ represent a random sample of size n from a distribution with pdf

$$f_\theta(x) = \begin{cases} \theta e^{-\theta x}, & x>0, \theta>0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the method of moments estimator for θ. Is this estimator unbiased?

[20 marks]

(c) Prove that the sum of the observations of a random sample of size n from a Poisson distribution having parameter θ, $0 < \theta < \infty$, is a sufficient statistic for θ.

[20 marks]

(d) Assume that X is a random variable from a distribution having pdf

$$f_\alpha(x) = \alpha e^{-\alpha x},\ x>0 \text{ and } \alpha>0.$$

(i) Is $2\alpha X$ a pivotal quantity? Explain.

(ii) Find the confidence coefficient for the random interval $(X, 3X)$ if this interval is a confidence interval for α.

(iii) What is the mathematical expectation of the length of the random interval in (ii)?

[40 marks]

(c) Biarkan X mempunyai taburan Poisson dengan parameter θ. Kebarangkalian priori terhadap Θ ialah $P(\Theta = 0.4) = \frac{2}{5}$ dan $P(\Theta = 0.3) = \frac{3}{5}$. Jika X = 4, cari yang berikut:

(i) Kebarangkalian posterior untuk $\Theta = 0.4$.

(ii) Kebarangkalian posterior untuk $\Theta = 0.3$.

[30 markah]

(d) Untuk fkk $f_\theta(x) = \theta(1-\theta)^x$, x = 0, 1, 2, ..., 0 < θ < 1, cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama θ berdasarkan suatu sampel rawak saiz n.

[20 markah]

4. *(a) Biarkan $X_1, X_2, ..., X_n$ sebagai suatu sampel rawak daripada taburan dengan fkk*

$$f_\theta(x) = \begin{cases} \theta\alpha x^{\alpha-1} eks(-\theta x^\alpha), & x > 0,\ \alpha \ diketahu \\ 0 & , di\ tempat\ lain \end{cases}.$$

Cari penganggar kebolehjadian maksimum bagi θ.

[20 markah]

(b) Biarkan $X_1, X_2, ..., X_n$ mewakili suatu sampel rawak saiz n daripada taburan dengan fkk

$$f_\theta(x) = \begin{cases} \theta e^{-\theta x}, & x > 0,\ \theta > 0 \\ 0 & , di\ tempat\ lain \end{cases}.$$

Cari penganggar kaedah momen untuk θ. Adakah penganggar ini saksama?

[20 markah]

(c) Buktikan bahawa hasil tambah cerapan-cerapan suatu sampel rawak saiz n daripada taburan Poisson dengan parameter θ, 0 < θ < ∞, adalah statistik cukup untuk θ.

[20 markah]

(d) Andaikan X adalah suatu pembolehubah rawak daripada taburan dengan fkk

$$f_\alpha(x) = \alpha e^{-\alpha x},\ \ x > 0 \ dan\ \ \alpha > 0.$$

(i) Adakah $2\alpha X$ suatu kuantiti pangsian?

(ii) Cari pekali keyakinan untuk selang rawak (X, 3X) jika selang ini adalah selang keyakinan bagi α.

(iii) Apakah jangkaan matematik untuk panjang selang rawak dalam (ii)?

[40 markah]

5. (a) Five independent samples, each of size *n*, are to be drawn from a normal distribution, where σ is known. For each sample, the interval $\left(\bar{x} - 0.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.06\frac{\sigma}{\sqrt{n}}\right)$ will be constructed. What is the probability that at least four of the intervals will contain the unknown mean μ?

[20 marks]

(b) Let $X_1, X_2, ..., X_n$ represent a random sample from a Poisson distribution with parameter λ, having a common pdf

$$f_\lambda(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, \ldots$$

Find the uniformly most powerful test of size α = 0.05 to test $H_0 : \lambda = 3$ vs. $H_1 : \lambda > 3$. Assume that the sample size, *n* = 25.

[20 marks]

(c) Assume that a single variable, *X* has a binomial, *b*(*n*, θ) distribution with probability mass function

$$f_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Find the likelihood ratio test for testing $H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$.

[30 marks]

(d) Let $X_1, X_2, ..., X_n$ be a random sample from a normal, $N(\mu, \sigma^2)$ distribution, having pdf

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty,$$

where σ^2 = 16. The hypotheses that we wish to test are $H_0 : \theta \le 18$ vs. $H_1 : \theta > 18$. The critical region for the test is $C = \left\{(x_1, x_2, ..., x_n) : \bar{x} > 18 + \frac{4}{\sqrt{n}}\right\}$.

(i) Find the power function of this test.

(ii) Sketch the graph of the power function versus θ when *n* = 25.

[30 marks]

5. (a) *Lima sampel tak bersandar, setiap dengan saiz n akan diambil daripada taburan normal, yang mana σ diketahui. Untuk setiap sampel, selang* $\left(\bar{x} - 0.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.06\frac{\sigma}{\sqrt{n}}\right)$ *akan dibina. Apakah kebarangkalian bahawa sekurang-kurangnya empat selang tersebut akan mengandungi min μ yang tidak diketahui?*

[20 markah]

(b) *Biarkan $X_1, X_2, \ldots, X_n$ mewakili suatu sampel rawak daripada taburan Poisson dengan parameter λ yang mempunyai fkk sepunya*

$$f_\lambda(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, \ldots$$

Cari ujian paling berkuasa secara seragam saiz α = 0.05 untuk menguji $H_0 : \lambda = 3$ lawan $H_1 : \lambda > 3$. Andaikan bahawa saiz sampel, n = 25.

[20 markah]

(c) *Andaikan bahawa pembolehubah tunggal, X mempunyai taburan binomial b(n, θ) dengan fungsi jisim kebarangkalian*

$$f_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Cari ujian nisbah kebolehjadian untuk menguji $H_0 : \theta \leq \theta_0$ lawan $H_1 : \theta > \theta_0$.

[30 markah]

(d) *Biarkan $X_1, X_2, \ldots, X_n$ sebagai suatu sampel rawak daripada taburan normal, $N(\mu, \sigma^2)$ yang mempunyai fkk*

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} eks\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty,$$

yang mana σ^2 = 16. Hipotesis yang kita ingin uji ialah $H_0 : \theta \leq 18$ lawan $H_1 : \theta > 18$. Rantau genting untuk ujian ini ialah $C = \left\{(x_1, x_2, \ldots, x_n) : \bar{x} > 18 + \frac{4}{\sqrt{n}}\right\}$.

(i) *Cari fungsi kuasa untuk ujian ini.*

(ii) *Lakarkan graf untuk fungsi kuasa lawan θ apabila n = 25.*

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x) = \frac{1}{N} I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q + pe^t\right)^n$
Geometri	$f(x) = pq^x I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t < 1$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}$, $t \neq 0$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\{-(x-\mu)^2 / 2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t + (\sigma t)^2 / 2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$, $t < \lambda$
Gama	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha$, $t < \lambda$
Khi Kuasa Dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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