UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MSG 388 – Mathematical Algorithms for Computer Graphics
[Algoritma Matematik untuk Grafik Komputer]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) State the definition of natural spline function.

   (b) Derive the Bézier form of a polynomial function $y(x) = (1+x)(1-x)^2$, where $x \in [-1, 1]$.

   (c) Given a rational cubic Bézier curve

$$\boldsymbol{R}(t) = \frac{w_0 \boldsymbol{C}_0 B_0^3(t) + w_1 \boldsymbol{C}_1 B_1^3(t) + w_2 \boldsymbol{C}_2 B_2^3(t) + w_3 \boldsymbol{C}_3 B_3^3(t)}{w_0 B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + w_3 B_3^3(t)}, \qquad t \in [0, 1]$$

   where $B_i^3(t)$ are the Bernstein polynomials of degree 3, the coefficients $\boldsymbol{C}_i \in \mathbb{R}^2$ are the Bézier points and the weights $w_i \geq 0$.

   (i) Suppose $\boldsymbol{C}_0$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{C}_3$ are non-collinear points and $w_1 = w_2 = w$ where $w > 0$, determine w_0 and w_3 such that $\boldsymbol{R}(t) = (1-t)\boldsymbol{C}_1 + t\boldsymbol{C}_2$, for any $t \in (0, 1)$.

   (ii) Derive $\boldsymbol{R}(t) = (x(t), y(t))$ which represents a circular arc $y = \sqrt{1-x^2}$, where $x \in [-\frac{1}{2}, 1]$.

[100 marks]

1. (a) *Nyatakan takrif fungsi splin asli.*

(b) *Terbitkan perwakilan Bézier suatu fungsi polinomial* $y(x) = (1+x)(1-x)^2$, *dengan* $x \in [-1, 1]$.

(c) *Diberi suatu lengkung Bézier kubik nisbah*

$$\mathbf{R}(t) = \frac{w_0 \mathbf{C}_0 B_0^3(t) + w_1 \mathbf{C}_1 B_1^3(t) + w_2 \mathbf{C}_2 B_2^3(t) + w_3 \mathbf{C}_3 B_3^3(t)}{w_0 B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + w_3 B_3^3(t)}, \qquad t \in [0, 1]$$

dengan $B_i^3(t)$ *polinomial Bernstein berdarjah* 3, *pekali* $\mathbf{C}_i \in \mathbb{R}^2$ *adalah titik Bézier dan pemberat* $w_i \geq 0$.

(i) *Andaikan* $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ *adalah titik-titik tak kolinear dan* $w_1 = w_2 = w$ *dengan* $w > 0$, *tentukan* w_0 *dan* w_3 *supaya* $\mathbf{R}(t) = (1-t)\mathbf{C}_1 + t\mathbf{C}_2$, *untuk sebarang* $t \in (0, 1)$.

(ii) *Terbitkan* $\mathbf{R}(t) = (x(t), y(t))$ *yang mewakili lengkok bulatan* $y = \sqrt{1-x^2}$, *dengan* $x \in [-\frac{1}{2}, 1]$.

[100 markah]

2. (a) Let $\boldsymbol{P}(u)$, $u \in [0, 1]$, be a cubic B-spline curve defined by a sequence of de Boor control points $(1, 1)$, $(1, 2)$, $(2, 2)$ and $(2, 1)$ over a knot vector $\boldsymbol{u} = (-3, -2, -1, 0, 1, 2, 3, 4)$.

   (i) Evaluate the derivative $\dfrac{\mathrm{d}^3 \boldsymbol{P}(u)}{\mathrm{d}u^3}$ at $u = 0.5$.

   (ii) Suppose a cubic B-spline curve $\bar{\boldsymbol{P}}(u)$ is defined over a knot vector $\bar{\boldsymbol{u}} = (-3, -2, -1, 0, 1, 1, 1, 2, 3, 4)$, find its de Boor points such that the curve $\bar{\boldsymbol{P}}(u)$ coincides with $\boldsymbol{P}(u)$.

(b) Find the de Boor points of a cubic uniform B-spline curve $\boldsymbol{P}(u)$ such that the Bézier form of $\boldsymbol{P}(u)$ is

$$\boldsymbol{P}(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1-u)^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix}u(1-u) + \begin{pmatrix} 3 \\ 0 \end{pmatrix}u^2, \qquad u \in [0, 1].$$

(c) Figure 1 shows a quadratic B-spline curve defined over the uniformly spaced knots $u_i = i$, integers $i \geq 0$, and its control polygon $\boldsymbol{D}_0\boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{D}_4\boldsymbol{D}_5$. Suppose the point $\boldsymbol{D}_2$ is duplicated from a single point towards triple points, describe the effect of multiple points $\boldsymbol{D}_2$ to the B-spline curve at segment $u \in [4, 5]$. Sketch the resulting B-spline curves and their control polygons when the double $\boldsymbol{D}_2$ and triple $\boldsymbol{D}_2$ are used.

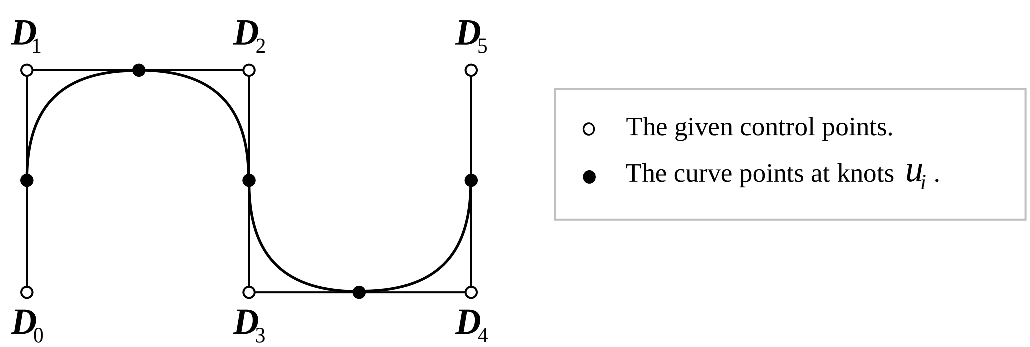


Figure 1

[100 marks]

2. (a) *Katakan* $\boldsymbol{P}(u)$, $u \in [0, 1]$, *ialah suatu lengkung splin-B kubik yang ditakrif oleh suatu jujukan titik kawalan de Boor* $(1, 1)$, $(1, 2)$, $(2, 2)$ *dan* $(2, 1)$ *pada suatu vektor simpulan* $\boldsymbol{u} = (-3, -2, -1, 0, 1, 2, 3, 4)$.

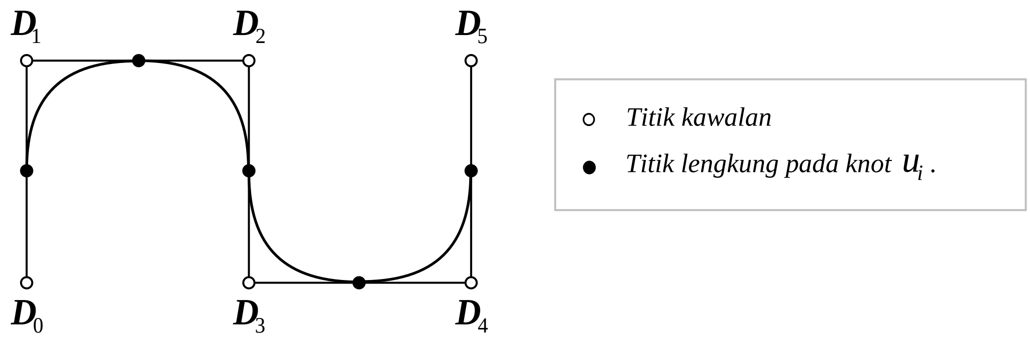
(i) *Nilaikan terbitan* $\dfrac{\mathrm{d}^3\boldsymbol{P}(u)}{\mathrm{d}u^3}$ *pada* $u = 0.5$.

(ii) *Andaikan suatu lengkung splin-B kubik* $\bar{\boldsymbol{P}}(u)$ *ditakrif pada suatu vektor simpulan* $\bar{\boldsymbol{u}} = (-3, -2, -1, 0, 1, 1, 1, 2, 3, 4)$, *cari titik-titik de Boor bagi* $\bar{\boldsymbol{P}}(u)$ *supaya lengkung ini sama dengan* $\boldsymbol{P}(u)$.

(b) *Cari titik-titik de Boor bagi suatu lengkung splin-B seragam kubik* $\boldsymbol{P}(u)$ *supaya perwakilan Bézier bagi* $\boldsymbol{P}(u)$ *ialah*

$$\boldsymbol{P}(u) = \begin{pmatrix}1\\0\end{pmatrix}(1-u)^2 + \begin{pmatrix}2\\1\end{pmatrix}u(1-u) + \begin{pmatrix}3\\0\end{pmatrix}u^2, \qquad u \in [0, 1].$$

(c) *Rajah 1 menunjukkan suatu lengkung splin-B kuadratik yang ditakrif pada simpulan seragam ruang* $u_i = i$, *integer* $i \geq 0$, *dan poligon kawalannya* $\boldsymbol{D}_0\boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{D}_4\boldsymbol{D}_5$. *Andaikan titik* $\boldsymbol{D}_2$ *diduplikasi daripada titik tunggal ke titik ganda tiga, huraikan kesan titik berganda* $\boldsymbol{D}_2$ *terhadap lengkung splin-B pada tembereng* $u \in [4, 5]$. *Lakar lengkung splin-B yang terhasil dan poligon kawalannya apabila duaan* $\boldsymbol{D}_2$ *dan tigaan* $\boldsymbol{D}_2$ *digunakan.*



Rajah 1

[100 markah]

3. (a) A biquadratic Bézier surface is defined by

$$P(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}\, B_i^2(x) B_j^2(y), \qquad 0\le x\le 1,\ \ 0\le y\le 1$$

where B_s^2, $s=0, 1, 2$, are the Bernstein polynomials of degree 2 and $c_{i,j}\in\mathbb{R}$ are the Bézier ordinates given as

$$c_{0,0}=1, \qquad c_{1,0}=3, \qquad c_{2,0}=1,$$
$$c_{0,1}=2, \qquad c_{1,1}=1, \qquad c_{2,1}=2,$$
$$c_{0,2}=1, \qquad c_{1,2}=3, \qquad c_{2,2}=1.$$

(i) Use the de Casteljau algorithm to evaluate the cross boundary derivative to surface P at $(x,y)=(1,\frac{1}{2})$.

(ii) Suppose the surface P is represented in parametric B-spline form as

$$\boldsymbol{P}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{D}_{i,j}\, N_i^3(u) N_j^3(v), \qquad 0\le u,v\le 1$$

where $\boldsymbol{D}_{i,j}\in\mathbb{R}^3$ are the de Boor points and N_s^3, $s=0, 1, 2$, are the normalized B-spline basis functions of order 3 defined over the knots $u_i=i$ and $v_i=i$, $i=-2, -1, \ldots, 3$. Determine all the $\boldsymbol{D}_{i,j}$.

(b) A quadratic Bézier triangle is defined by

$$\boldsymbol{S}(u,v,w)=\sum_{\substack{i,j,k\ge 0\\ i+j+k=2}}\boldsymbol{C}_{i,j,k}\, B_{i,j,k}^2(u,v,w), \qquad 0\le u,v,w\le 1, \quad u+v+w=1$$

where $B_{i,j,k}^2(u,v,w)$ are the generalised Bernstein polynomials of degree 2 and $\boldsymbol{C}_{i,j,k}\in\mathbb{R}^3$ are the Bézier points given as

$$\boldsymbol{C}_{2,0,0}=(2,5,2),$$
$$\boldsymbol{C}_{1,1,0}=(1\tfrac{1}{2},3,2), \qquad \boldsymbol{C}_{1,0,1}=(3,3\tfrac{1}{2},2),$$
$$\boldsymbol{C}_{0,2,0}=(1,1,3), \qquad \boldsymbol{C}_{0,1,1}=(2\tfrac{1}{2},1\tfrac{1}{2},2), \qquad \boldsymbol{C}_{0,0,2}=(4,2,1).$$

(i) If $\boldsymbol{P}=(3,2,z)$ is a point on the surface $\boldsymbol{S}(u,v,w)$, evaluate the value z.

(ii) Evaluate the directional derivative of $\boldsymbol{S}(u,v,w)$ with a vector $\boldsymbol{d}=(-\frac{1}{2}, 1, -\frac{1}{2})$ at $(u, v, w)=(\frac{1}{3}, 0, \frac{2}{3})$.

[100 marks]

3. (a) *Suatu permukaan Bézier bikuadratik ditakrif oleh*

$$P(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2} c_{i,j}\, B_i^2(x) B_j^2(y), \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

dengan $B_s^2,\ s=0,\ 1,\ 2$, *polinomial Bernstein berdarjah* 2 *dan* $c_{i,j}\in\mathbb{R}$ *adalah ordinat Bézier yang diberikan sebagai*

$c_{0,0}=1,\qquad c_{1,0}=3,\qquad c_{2,0}=1,$

$c_{0,1}=2,\qquad c_{1,1}=1,\qquad c_{2,1}=2,$

$c_{0,2}=1,\qquad c_{1,2}=3,\qquad c_{2,2}=1.$

(i) *Guna algoritma de Casteljau untuk menilai terbitan sempadan silang kepada permukaan* P *pada* $(x,y)=(1,\ \frac{1}{2})$.

(ii) *Andaikan permukaan* P *diwakili dalam bentuk splin-B berparameter*

$$\boldsymbol{P}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2} \boldsymbol{D}_{i,j}\, N_i^3(u) N_j^3(v), \qquad 0 \le u,v \le 1$$

dengan $\boldsymbol{D}_{i,j}\in\mathbb{R}^3$ *ialah titik de Boor dan* $N_s^3,\ s=0,\ 1,\ 2$, *adalah fungsi asas splin-B ternormal berperingkat* 3 *yang ditakrif pada simpulan* $u_i=i$ *dan* $v_i=i$, $i=-2,\ -1,\ \ldots,\ 3$. *Tentukan semua titik* $\boldsymbol{D}_{i,j}$.

(b) *Suatu segi tiga Bézier kuadratik ditakrif oleh*

$$\boldsymbol{S}(u,v,w)=\sum_{\substack{i,j,k\ge 0\\ i+j+k=2}} \boldsymbol{C}_{i,j,k}\, B_{i,j,k}^2(u,v,w), \qquad 0\le u,v,w \le 1, \quad u+v+w=1$$

dengan $B_{i,j,k}^2(u,v,w)$ *ialah polinomial Bernstein teritlak berdarjah* 2 *dan* $\boldsymbol{C}_{i,j,k}\in\mathbb{R}^3$ *adalah titik Bézier yang diberikan sebagai*

$\boldsymbol{C}_{2,0,0}=(2,5,2),$

$\boldsymbol{C}_{1,1,0}=(1\tfrac{1}{2},3,2),\qquad \boldsymbol{C}_{1,0,1}=(3,3\tfrac{1}{2},2),$

$\boldsymbol{C}_{0,2,0}=(1,1,3),\qquad \boldsymbol{C}_{0,1,1}=(2\tfrac{1}{2},1\tfrac{1}{2},2),\qquad \boldsymbol{C}_{0,0,2}=(4,2,1).$

(i) *Jika* $\boldsymbol{P}=(3,2,z)$ *ialah suatu titik pada permukaan* $\boldsymbol{S}(u,v,w)$, *nilaikan* z.

(ii) *Nilaikan terbitan berarah* $\boldsymbol{S}(u,v,w)$ *dengan vektor* $\boldsymbol{d}=(-\frac{1}{2},1,-\frac{1}{2})$ *pada* $(u,\ v,\ w)=(\frac{1}{3},\ 0,\ \frac{2}{3})$.

[100 markah]

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