UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MSG 327 – Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of THIRTEEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TIGA BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]

1. (a) Give a definition of mathematical model and discuss briefly on the procedure that is used in constructing the model.

(b) Suppose that a healthy population of some species is growing in a limited environment and that the current population P_0 is fairly close to the carrying capacity M_0. You might imagine a population of fish living in a freshwater lake in wilderness area. Suddenly, a catastrophe contaminates the lake and destroys a significant part of the food and oxygen on which the fish depend. The result is a new environment with a carrying capacity M_1 considerably less than M_0 and, in fact, less than the current population P_0. Starting at some time before the catastrophe, sketch a "before-and-after" curve that shows how the fish population responds to the change in the environment.

(c) Suppose a large lake that was formed by damming a river holds initially 150 million gallons of water. Because a nearby agricultural field was sprayed with a pesticide, the water has become contaminated. The concentration of the pesticide has been measured and is equal to 40 ppm (parts per million), or 40×10^{-6}. The river continues to flow into the lake at a rate of 300 gal/min. The river is only slightly contaminated with pesticide and has a concentration of 10 ppm. The flow of water over the dam can be controlled and is set at 450 gal/min. Assume that no additional spraying causes the lake to become even more contaminated. How long will it be before the water reaches an acceptable level of concentration equal to 15 ppm?

[100 marks]

1. *(a)* *Berikan definisi model matematik dan bincangkan secara ringkas prosedur yang digunakan dalam pembangunan model tersebut.*

   (b) *Andaikan populasi sejahtera satu spesis membiak di dalam persekitaran terbatas dengan populasi semasanya* P_0 *menghampiri kapasiti bawaan* M_0. *Bayangkan populasi ikan di dalam tasik air tawar di hutan. Bencana alam yang berlaku telah mencemari tasik itu dan memusnahkan bahagian penting makanan dan oksigen yang diperlukan oleh ikan. Maka persekitaran baru terbentuk dengan kapasiti bawaan* M_1 *kurang daripada* M_0 *dan populasi semasa* P_0. *Bermula pada masa sebelum bencana berlaku, lakarkan lengkung "sebelum-dan-selepas" yang menunjukkan bagaimana populasi ikan bertindak balas dengan perubahan persekitaran.*

   (c) *Andaikan sebuah tasik luas yang terbentuk dari empangan sungai, yang pada mulanya mengandungi 150 juta gelen air. Kemudian airnya dicemari oleh racun serangga yang disemburkan di kawasan pertanian berdekatan. Kepekatan racun serangga itu telah diukur dan ia bersamaan dengan 40 ppm (bahagian per juta), atau* 40×10^{-6}. *Air sungai terus mengalir ke dalam tasik tersebut pada kadar 300 gal/min dengan kepekatan racun serangga yang sedikit iaitu 10 ppm. Pengaliran air ke dalam empangan ditentukan dengan kadar 450 gal/min. Anggap bahawa tiada semburan racun serangga tambahan yang akan mencemarkan lagi tasik. Berapa lamakah masa yang diperlukan bagi mencapai kepekatan pada tahap kesesuaian 15 ppm?*

[100 markah]

2. A patient is given a dosage Q of a drug at regular intervals of time T. The concentration of the drug in the blood has been shown experimentally to obey the law

$$\frac{dC}{dt} = -ke^{C} \qquad (1)$$

(a) If the first dose is administered at $t = 0$ hr, show that after T hr have elapsed, the residual

$$R_1 = -\ln\ kT + e^{-Q} \qquad (2)$$

remains in the blood.

(b) Assuming an instantaneous rise in concentration whenever the drug is administered, show that after the second dose and T hr have elapsed again, the residual

$$R_2 = -\ln\left[kT\ 1 + e^{-Q}\ + e^{-2Q}\right] \qquad (3)$$

remains in the blood.

(c) Show that the limiting value R of the residual concentrations for doses of Q mg/ml repeated at intervals of T hr is given by the formula

$$R = -\ln\frac{kT}{1-e^{-Q}}\,. \qquad (4)$$

(d) Assuming the drug is ineffective below a concentration L and harmful above some higher concentration H, show that the dose schedule T for a safe and effective concentration of the drug in the blood satisfies the formula

$$T = \frac{1}{k}\ e^{-L} - e^{-H} \qquad (5)$$

where k is a positive constant.

[80 marks]

2. *Seorang pesakit diberi Q dos ubat pada selang masa T yang tetap. Kepekatan ubat di dalam darah ditunjukkan secara eksperimen telah mematuhi hukum*

$$\frac{dC}{dt} = -ke^{C} \qquad (1)$$

(a) *Jika dos pertama diberi pada $t = 0$ jam, tunjukkan bahawa selepas T jam berlalu, baki yang tertinggal di dalam darah adalah*

$$R_1 = -\ln\ kT + e^{-Q}\ . \qquad (2)$$

(b) *Anggapkan kepekatan meningkat secara mendadak apabila ubat diberi, tunjukkan bahawa selepas dos kedua dan T jam berlalu lagi, baki yang tertinggal di dalam darah adalah*

$$R_2 = -\ln\left[kT\ 1 + e^{-Q}\ + e^{-2Q}\right]. \qquad (3)$$

(c) *Tunjukkan bahawa nilai terbatas R bagi kepekatan baki untuk Q mg/ml dos berulang pada selang masa T jam diberikan oleh formula*

$$R = -\ln\frac{kT}{1 - e^{-Q}}\,. \qquad (4)$$

(d) *Andaikan ubat tidak berkesan apabila kurang daripada kepekatan L dan merbahaya apabila lebih daripada kepekatan H, tunjukkan bahawa penjadualan dos T untuk kepekatan ubat yang selamat dan berkesan di dalam darah memenuhi formula*

$$T = \frac{1}{k}\ e^{-L} - e^{-H} \qquad (5)$$

dengan k adalah pemalar positif.

[80 markah]

3. Consider two species (x and y) whose survival depends on their mutual cooperation. Let's take as an example a species of bee that feeds primarily on the nectar of one plant species and simultaneously pollinates that plant. One simple model of this mutualism is given by the autonomous system

$$\frac{dx}{dt} = ax - bxy \tag{6}$$

$$\frac{dy}{dt} = -my + nxy \tag{7}$$

where the initial conditions $x(0) = x_0$ and $y(0) = y_0$.

(a) What assumptions are implicitly being made about the growth of each species in the absence of cooperation?

(b) Interpret the constants a, b, m and n in terms of the physical problem.

(c) What are the equilibrium levels?

(d) Perform a graphical analysis and indicate the trajectory directions in the phase plane.

(e) Find an analytic solution and sketch typical trajectories in the phase plane.

(f) Interpret the outcomes predicted by your graphical analysis. Do you believe the model is realistic? Why?

[120 marks]

3. *Pertimbangkan dua spesis (x dan y) yang kemandirian mereka bersandar kepada kerjasama saling antara mereka. Contohnya spesis lebah yang menghisap madu daripada satu spesis pokok dan secara serentak mendebungakan pokok itu. Satu model ringkas kesalingan ini diberikan oleh sistem berautonomi*

$$\frac{dx}{dt} = ax - bxy \tag{6}$$

$$\frac{dy}{dt} = -my + nxy \tag{7}$$

dengan syarat awal $x(0) = x_0$ *dan* $y(0) = y_0$.

(a) *Apakah anggapan-anggapan yang perlu dibuat mengenai pertumbuhan setiap spesis yang terlibat dengan kerjasama saling ini?*

(b) *Tafsirkan pemalar* a, b, m *dan* n *dalam bentuk masalah fizikal.*

(c) *Apakah aras keseimbangannya?*

(d) *Lakukan analisis bergraf dan tunjukkan arah-arah trajektori dalam satah fasa.*

(e) *Dapatkan penyelesaian analitiknya dan lakarkan trajektori-trajektori tipikal dalam satah fasa.*

(f) *Tafsirkan hasil yang diperoleh berdasarkan analisis bergraf anda. Adakah model anda itu realistik? Kenapa?*

[120 markah]

4. (a) Find the first three approximations x_1, y_1 , x_2, y_2 and x_3, y_3 using Euler's method for the predator-prey system

$$\frac{dx}{dt} = 2x + 3y \tag{8}$$

$$\frac{dy}{dt} = 3x + 2y \tag{9}$$

subject to the initial conditions $x(0) = 1$ and $y(0) = 1$ starting at $t(0) = 0$, with $\Delta t = \frac{1}{4}$.

(i) Repeat your calculations for $\frac{\Delta t}{2}$.

(ii) Compare your approximations with the values of the given analytical solution $x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$ and $y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$. What can you say about the comparison?

(b) The following system is a predator-prey model in which human harvesting occurs for both species. Use Euler's method with step size $\Delta t = 1$ over $0 \leq t \leq 4$ to numerically solve

$$\frac{dx}{dt} = x - xy - \frac{3}{4}y \tag{10}$$

$$\frac{dy}{dt} = xy - y - \frac{3}{4}x \tag{11}$$

subject to $x(0) = \frac{1}{2}$ and $y(0) = 1$.

Then use Euler's method to solve the above model (Eqs. (10) and (11)) without harvesting. Compare and discuss the differences in your solutions to the two models.

[100 marks]

4. (a) *Dapatkan tiga penghampiran pertama* x_1, y_1 , x_2, y_2 *dan* x_3, y_3 *menggunakan kaedah Euler untuk sistem pemangsa-mangsa*

$$\frac{dx}{dt} = 2x + 3y \qquad (8)$$

$$\frac{dy}{dt} = 3x + 2y \qquad (9)$$

tertakluk kepada syarat awal $x(0) = 1$ *dan* $y(0) = 1$ *bermula pada* $t(0) = 0$, *dengan* $\Delta t = \frac{1}{4}$.

(i) *Ulangi kiraan anda untuk* $\frac{\Delta t}{2}$.

(ii) *Bandingkan pengahampiran-penghampiran yang diperoleh dengan nilai-nilai penyelesaian analitik* $x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$ *dan* $y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$. *Apa yang boleh diperkatakan tentang perbandingan tersebut?*

(b) *Berikut ialah sistem model pemangsa-mangsa yang melibatkan proses penuaian bagi kedua-dua spesis. Gunakan kaedah Euler dengan saiz selang* $\Delta t = 1$ *dalam* $0 \le t \le 4$ *bagi menyelesaikan secara berangka*

$$\frac{dx}{dt} = x - xy - \frac{3}{4}y \qquad (10)$$

$$\frac{dy}{dt} = xy - y - \frac{3}{4}x \qquad (11)$$

tertakluk kepada $x(0) = \frac{1}{2}$ *dan* $y(0) = 1$.

Seterusnya, gunakan kaedah Euler untuk menyelesaikan model di atas (Pers. (10) and (11)) tanpa proses penuaian. Banding dan bincangkan perbezaan-perbezaan dalam penyelesaian dua model tersebut.

[100 markah]

5. Consider launching a satellite into orbit using a single-stage rocket. The rocket is continuously losing mass, which is being propelled away from it at significant speeds. We are interested in predicting the maximum speed the rocket can attain.

(a) Assume the rocket mass m is moving with speed v. In a small increment of time Δt it loses a small mass Δm_p, which leaves the rocket with speed u in a direction opposite to v. Here Δm_p is the small propellant mass. The resulting speed of the rocket is $v+\Delta v$. Neglect all external forces (gravity, atmospheric drag, etc.) and assume Newton's second law of motion:

$$\text{force} = \frac{d}{dt}(\text{momentum of system}) \tag{12}$$

where momentum is mass time velocity. Derive the model

$$\frac{dv}{dt} = \left(\frac{-c}{m}\right)\frac{dm}{dt} \tag{13}$$

where $c = u + v$ is the relative exhaust speed (speed of the burnt gases relative to the rocket).

(b) Assume that initially, at time $t = 0$, the velocity $v = 0$ and the mass of the rocket is $m = M + P$, where P is the mass of the payload satellite and $M = \varepsilon M + (1-\varepsilon)M$ $(0 < \varepsilon < 1)$ is the initial fuel mass εM plus the mass $(1-\varepsilon)M$ of the rocket casings and instruments. Solve the model in part (a) to obtain the speed

$$v = -c \ln \frac{m}{M+P}. \tag{14}$$

(c) Show that when all the fuel is burned, the speed of the rocket is given by

$$v_f = -c \ln\left[1 - \frac{\varepsilon}{1+\beta}\right] \tag{15}$$

where $\beta = P/M$ is the ratio of the payload mass to the rocket mass.

5. *Pertimbangkan pelancaran satelit ke dalam orbit menggunakan roket satu-peringkat. Jisim roket itu berkurang secara berterusan dan jisim yang hilang itu terdorong jauh daripada roket dengan kelajuan tertentu. Kita berminat dalam menentukan halaju maksimum yang mampu dicapai oleh roket tersebut.*

(a) *Anggapkan jisim roket* m *bergerak dengan halaju* v. *Dalam selang masa* Δt, *roket tersebut mengalami pengurangan jisim sebanyak* Δm_p *dengan halaju selepas pengurangan jisim ialah* u *yang arahnya adalah bertentangan dengan* v. Δm_p *ialah jisim yang terdorong jauh daripada roket dengan kuantiti yang kecil. Halaju roket yang terhasil ialah* $v+\Delta v$. *Abaikan semua daya luar (graviti, tarikan atmosferik, dan lain-lain) dan anggapkan ia memenuhi Hukum Gerakan Newton kedua:*

$$daya = \frac{d}{dt}(sistem\ momentum) \qquad (12)$$

dengan momentum ialah jisim darab halaju. Terbitkan model

$$\frac{dv}{dt} = \left(\frac{-c}{m}\right)\frac{dm}{dt} \qquad (13)$$

dengan $c = u + v$ *ialah halaju buangan relatif (halaju pembakaran gas relatif kepada roket).*

(b) *Anggapkan pada awalnya iaitu pada masa* $t = 0$, *halaju* $v = 0$ *dan jisim roket ialah* $m = M + P$, *dengan* P *ialah jisim muatan satelit dan* $M = \varepsilon M + (1-\varepsilon)M \quad (0 < \varepsilon < 1)$ *ialah nilai awal jisim bahan api* εM *ditambah dengan jisim bingkai dan peralatan roket* $(1-\varepsilon)M$. *Selesaikan model dalam bahagian (a) bagi memperoleh halaju*

$$v = -c\ln\frac{m}{M+P}. \qquad (14)$$

(c) *Tunjukkan bahawa apabila semua bahan api digunakan, halaju roket diberikan oleh*

$$v_f = -c\ln\left[1 - \frac{\varepsilon}{1+\beta}\right] \qquad (15)$$

dengan $\beta = P/M$ *ialah kadar bagi jisim muatan dan jisim roket.*

(d) Find v_f if $c = 3$ km/sec, $\varepsilon = 0.8$ and $\beta = 1/100$. (These are typical values in satellite launchings.)

(e) Suppose scientists plan to launch a satellite in circular orbit h km above the earth's surface. Assume that the gravitational pull toward the center of the earth is given by Newton's inverse square law of attraction:

$$\frac{\gamma m M_e}{(h+R_e)^2}$$

where γ is the universal gravitational constant, m is the mass of the satellite, M_e is the earth's mass and R_e is the radius of the earth. Assume that this force must be balanced by the centrifugal force $mv^2/(h+R_e)$, where v is the speed of the satellite. What speed must be attained by a rocket to launch a satellite into an orbit 100 km above the earth's surface? From your computation in part (d), can a single-stage rocket launch a satellite into an orbit of that height?

[100 marks]

(d) *Dapatkan* v_f *jika* $c = 3$ *km/saat,* $\varepsilon = 0.8$ *dan* $\beta = 1/100$. *(Ini ialah nilai-nilai tipikal dalam pelancaran satelit)*

(e) *Andaikan ahli-ahli sains merancang untuk melancarkan satelit ke dalam orbit membulat yang jaraknya* h *km dari permukaan bumi. Anggapkan bahawa tarikan graviti ke arah pusat bumi diberikan oleh Hukum Tarikan Songsangan kuasa dua Newton:*

$$\frac{\gamma m M_e}{(h + R_e)^2}$$

dengan γ *ialah pemalar graviti sejagat,* m *ialah jisim satelit,* M_e *ialah jisim bumi dan* R_e *ialah jejari bumi. Anggapkan bahawa daya ini mesti seimbang dengan daya emparan* $mv^2/(h + R_e)$, *dengan* v *ialah halaju satelit. Berapakah halaju yang perlu dicapai oleh roket untuk melancarkan satelit ke dalam orbit yang jaraknya 100 km dari permukaaan bumi. Berdasarkan kiraan anda dalam bahagian (d), bolehkah roket satu-peringkat digunakan untuk melancarkan satelit ke dalam orbit dengan jarak yang dinyatakan sebelum ini?*

[100 markah]

- ooo O ooo -