UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Consider the wave equation $u_{tt} = u_{xx}$. Write down the centered time, centered space finite difference scheme for this equation.

(b) Analyze the stability of the scheme in (a) using the Fourier (von Neumann) method.

(c) State the Gerschgorin Circle Theorem.

[100 marks]

2. (a) Consider the equation $u_t + 2u_x = 0$. Write down the forward time, centred space finite difference scheme for this equation.

(b) Given the following initial and boundary conditions:

$$u(x,0) = \sin \pi x,\ 0 \le x \le 1$$
$$u(0,t) = 0 \quad \forall t > 0$$
$$u(1,t) = 0 \quad \forall t > 0.$$

Solve the equation in 2 (a) using the forward time, centred space scheme with $\Delta x = 0.25, \Delta t = 0.5$. Obtain the solutions at $x = 0.25, 0.5, 0.75$ for $t = 1$.

(c) State the Lax Equivalence Theorem.

[100 marks]

1. (a) *Pertimbangkan persamaan gelombang* $u_{tt} = u_{xx}$. *Tulis skema beza terhingga pusat terhadap masa, beza pusat terhadap ruang bagi persamaan ini.*

   (b) *Analisis skema dalam (a) dengan menggunakan kaedah Fourier (von Neumann).*

   (c) *Nyatakan Teorem Bulatan Gerschgorin*

[100 markah]

2. (a) *Pertimbangkan persamaan* $u_t + 2u_x = 0$. *Tulis skema beza terhingga ke depan terhadap masa, beza pusat terhadap ruang bagi persamaan ini.*

   (b) *Diberi nilai awal dan sempadan berikut:*

$$u(x,0) = \sin \pi x \,,\ 0 \le x \le 1$$
$$u(0,t) = 0 \quad \forall t > 0$$
$$u(1,t) = 0 \quad \forall t > 0.$$

   Selesaikan persamaan dalam 2 (a) dengan menggunakan skema beza kedepan terhadap masa, beza pusat terhadap ruang dengan $\Delta x = 0.25, \Delta t = 0.5$. *Dapatkan penyelesaian pada* $x = 0.25, 0.5, 0.75$ *untuk* $t = 1$.

   (c) *Nyatakan Teorem Kesetaraan Lax.*

[100 markah]

3. (a) Suppose that the coefficient matrix A of the system $A\underline{u} = \underline{b}$ is

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

(i) Find the eigen values of the Jacobi iteration matrix G_J.

(ii) Compute the spectral radius of G_J.

(iii) In solving this system, what is the optimum relaxation parameter ω for the point SOR method?

(iv) Let P be the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that A is two-cyclic, i.e. A satisfies the property $PAP^T = \begin{bmatrix} D_1 & F \\ G & D_2 \end{bmatrix}$, where D_1 and D_2 are square diagonal matrices and F and G are rectangular matrices.

(b) Consider the system

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \underline{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

(i) In solving this system, show that the SOR iterative formula is as follows:

$$\underline{u}^{(k+1)} = \begin{bmatrix} 1-\omega & -\omega/2 \\ \omega(\omega-1)/3 & (\omega^2/6-\omega+1) \end{bmatrix} \underline{u}^{(k)} + \begin{bmatrix} 3\omega \\ -\omega^2-2\omega/3 \end{bmatrix}.$$

(ii) Find the spectral radius of the Gauss-Seidel iteration matrix.

[100 marks]

3. (a) *Katakan matriks koefisien A bagi sistem* $A\underline{u} = \underline{b}$ *adalah*

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

(i) *Cari nilai-nilai eigen bagi matriks lelaran Jacobi* G_J.

(ii) *Kirakan jejari spektrum bagi* G_J.

(iii) *Dalam menyelesaikan sistem ini, apakah parameter pengenduran optimum* ω *bagi kaedah titik SOR?*

(iv) *Biarkan P merupakan matriks permutasi*

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Tunjukkan bahawa A adalah berkitar-dua, i.e. A memenuhi sifat $PAP^T = \begin{bmatrix} D_1 & F \\ G & D_2 \end{bmatrix}$, *di mana* D_1 *dan* D_2 *adalah matriks pepenjuru segiempat sama dan F dan G adalah matriks segiempat tepat.*

(b) *Pertimbangkan sistem*

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \underline{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

(i) *Dalam menyelesaikan sistem ini, tunjukkan bahawa rumus lelaran SOR adalah seperti berikut:*

$$\underline{u}^{(k+1)} = \begin{bmatrix} 1-\omega & -\omega/2 \\ \omega(\omega-1)/3 & (\omega^2/6-\omega+1) \end{bmatrix} \underline{u}^{(k)} + \begin{bmatrix} 3\omega \\ -\omega^2 - 2\omega/3 \end{bmatrix}.$$

(ii) *Cari jejari spektrum bagi matriks lelaran Gauss-Seidel.*

[100 markah]

4. (a) Consider the following elliptic problem

$$\nabla^2 u = x^2, \qquad 0 < x, y < 1$$
$$u(0,y) = u(1,y) = 0, \qquad 0 \le y \le 1$$
$$u(x,0) = u(x,1) = 0, \qquad 0 \le \mathrm{x} \le 1$$

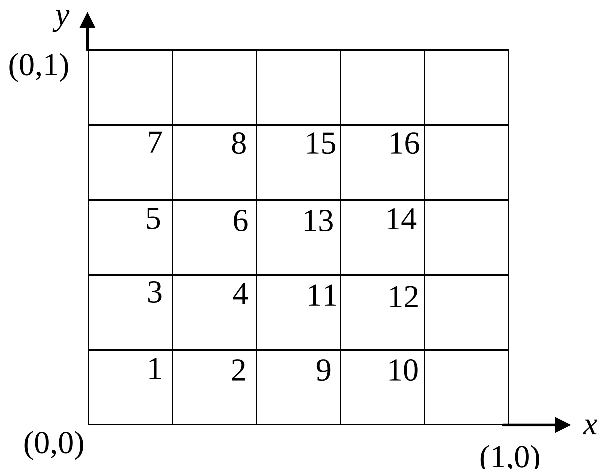


Figure 1

(i) Using the five-point finite difference approximation, formulate the **block** matrix system A$\underline{u}$ = $\underline{b}$ of the two-line iterative method taken in the ordering and mesh size as shown in Figure 1.

(ii) What is the estimated optimum relaxation parameter ω_b and the spectral radius of the two line SOR (S.2.L.O.R) iteration matrix, $\rho(L_{\omega_b}^{2-line})$, for the shown mesh size?

(iii) What is the approximate theoretical number of iterations you would expect to get if the two-line SOR method is used for mesh size n = 35 and tolerance $\varepsilon = 10^{-6}$.

(iv) What is the rate of convergence $R_\infty(L_{\omega_b}^{2-line})$ of this method for the mesh size n = 35? What is the rate of convergence $R_\infty(L_{\omega_b})$ of the point SOR for the same mesh size? What can you conclude about their convergence rates?

(b) Suppose that A is a $m \times m$ symmetric, positive definite and tridiagonal matrix, whose Jacobi iterative matrix is

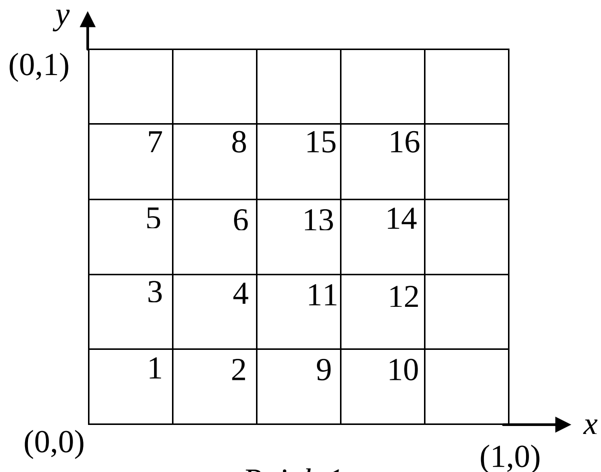
$$G = g_{ij}\ {}_{i,j=1,2,\ldots,m} = \begin{cases} \frac{1}{4} & \text{if } j = i+1, \quad i = 1,2,\ldots m-1 \\ 0.125 & \text{if } j = i-1, \quad i = 2,3,4\ldots,m \\ 0 & \text{if } j = i, i = 1,2,3,\ldots,m \quad \text{and elsewhere} \end{cases}$$

What is the spectral radius of this Jacobi iterative matrix when m = 67?

[100 marks]

4. (a) *Pertimbangkan masalah eliptik berikut*

$$\nabla^2 u = x^2, \qquad 0 < x, y < 1$$
$$u(0,y) = u(1,y) = 0, \qquad 0 \le y \le 1$$
$$u(x,0) = u(x,1) = 0, \qquad 0 \le x \le 1$$



Rajah 1

(i) *Dengan menggunakan penganggaran beza terhingga lima-titik, formulasikan sistem matriks* ***blok*** *A$\underline{u}$ = $\underline{b}$ bagi kaedah lelaran dua-garis yang diambil dalam tertib dan saiz mesh seperti yang ditunjukkan dalam Rajah 1.*

(ii) *Apakah parameter pengenduran optimum teranggar ω_b dan jejari spektrum bagi matriks lelaran SOR dua-garis (S.2.L.O.R), $\rho(L_{\omega_b}^{2-line})$, bagi saiz mesh yang ditunjukkan?*

(iii) *Apakah anggaran bilangan lelaran berteori yang anda jangka didapati sekiranya kaedah SOR dua-garis digunakan bagi saiz mesh n = 35 dan tolerans $\varepsilon = 10^{-6}$.*

(iv) *Apakah kadar penumpuan $R_\infty(L_{\omega_b}^{2-line})$ bagi kaedah ini untuk saiz mesh n = 35? Apakah kadar penumpuan $R_\infty(L_{\omega_b})$ bagi kaedah titik SOR untuk saiz mesh yang sama? Apakah yang anda boleh simpulkan mengenai kadar penumpuan mereka?*

(b) *Katakan A ialah matriks (mxm) yang simetri, tentu positif dan tiga pepenjuru, yang matriks lelaran Jacobi nya ialah*

$$G = g_{ij}\ {}_{i,j=1,2,\ldots,m} = \begin{cases} \dfrac{1}{4} & \text{jika } j = i+1, \quad i = 1,2,\ldots m-1 \\ 0.125 & \text{jika } j = i-1, \quad i = 2,3,4\ldots,m \\ 0 & \text{jika } j = i, i = 1,2,3,\ldots,m \quad \text{dan lain-lain} \end{cases}$$

Apakah jejari spektrum bagi matriks lelaran Jacobi ini apabila m = 67?

[100 markah]

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