UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MAT 517 – Computational Linear Algebra and Function Approximation
[Aljabar Linear Pengkomputeran dan Penghampiran Fungsi]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **SEVEN** pages of printed material before you begin the examination.

[*Sila pastikan bahawa kertas peperiksaan ini mengandungi* ***TUJUH*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*]

Instructions: Answer **all four** [4] questions.

[**Arahan:** Jawab **semua empat** [4] soalan.]

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Let matrix $\mathbf{A}=\begin{bmatrix} 0.001 & 1 \\ 1 & 1 \end{bmatrix}$.

   (a) Use Gaussian elimination to factor $\mathbf{A}$ into the product $\mathbf{LU}$ where $\mathbf{L}$ is a lower triangular matrix whose diagonal entries are 1, and, $\mathbf{U}$ is an upper triangular matrix.

   (b) Use the $\mathbf{LU}$ factorization in part (a) to solve the linear equation $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{b}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   (c) Use Householder transformation to produce $\mathbf{QR}$ factorization of $\mathbf{A}$. Use this factorization to solve the same system as in part (b).

   (d) Compare the result in part (b) and (c) with the exact solution to the problem.

   (e) Is your solution in part (c) more accurate than the result which you have obtained earlier in part (b). Why?

[100 marks]

2. Consider the problem of finding a vector $\mathbf{x}\in\mathbb{R}^2$ that minimizes $\|\mathbf{b}-\mathbf{Ax}\|_2^2$, where $\mathbf{b}=\begin{pmatrix} 4 \\ -5 \\ 5 \end{pmatrix}$, and $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 2 & 4 \end{bmatrix}$ has a singular value decomposition (SVD) of the form $\mathbf{A}=\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U}$ is a 3×3 orthogonal matrix, $\mathbf{\Sigma}$ is a 3×2 "diagonal matrix," and $\mathbf{V}$ is a 2×2 orthogonal matrix.

   (a) Find matrices $\mathbf{U}, \mathbf{\Sigma}$ and $\mathbf{V}$.

   (b) Find the pseudoinverse of **A** (which is also denoted by $\mathbf{A}^{-1}$).

   (c) Find the least squares solution to **Ax** = **b**.

[100 marks]

1. *Andaikan matriks* $\mathbf{A} = \begin{bmatrix} 0.001 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) *Guna kaedah penghapusan Gauss untuk memfaktor* $\mathbf{A}$ *kepada hasil darab* $\mathbf{LU}$ *dengan* $\mathbf{L}$ *matriks segi tiga bawah yang mempunyai pemasukan pepenjuru 1, dan* $\mathbf{U}$ *matriks segi tiga atas.*

(b) *Guna pemfaktoran* $\mathbf{LU}$ *dalam bahagian (a) untuk menyelesaikan persamaan linear* $\mathbf{Ax} = \mathbf{b}$, *dengan* $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) *Guna transformasi Householder untuk menghasilkan pemfaktoran* $\mathbf{QR}$ *kepada* $\mathbf{A}$. *Guna pemfaktoran ini untuk menyelesaikan sistem yang sama seperti yang terdapat di bahagian (b).*

(d) *Bandingkan keputusan dalam bahagian (b) dan (c) dengan penyelesaian tepat masalah berkenaan.*

(e) *Adakah penyelesaian yang anda peroleh di bahagian (c) lebih jitu daripada keputusan yang anda telah peroleh lebih awal dalam bahagian (b). Kenapa?*

[100 markah]

2. *Pertimbangkan masalah mencari suatu vektor* $\mathbf{x} \in \mathbb{R}^2$ *yang meminimumkan* $\|\mathbf{b} - \mathbf{Ax}\|_2^2$, *dengan* $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \\ 5 \end{pmatrix}$, *dan* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 2 & 4 \end{bmatrix}$ *mempunyai penghuraian nilai singular (PNS) berbentuk* $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$, *dengan* $\mathbf{U}$ *matriks ortogon* 3×3, Σ *"matriks pepenjuru"* 3×2, *dan* $\mathbf{V}$ *matriks ortogon* 2×2.

(a) *Cari matriks* $\mathbf{U}, \Sigma$ dan $\mathbf{V}$.

(b) *Cari pseudo songsang* ***A*** *(yang juga ditandai sebagai* $\mathbf{A}^{-1}$).

(c) *Cari penyelesaian kuasa dua terkecil kepada* ***Ax*** = ***b***.

[100 markah]

3. (a)

x	$f(x)$	$f'(x)$
0.30	0.29552	0.95534
0.32	0.31457	0.94924
0.35	0.34290	0.93937

[i] Use the above values of x and five-digit rounding arithmetic for $f(x)$ and $f'(x)$ to construct the Hermite interpolating polynomial, $H_5(x)$.

[ii] Then, use the constructed polynomial in part [i] to approximate $f(0.34)$.

(b) Let the partition $x_0 = 0, x_1 = 0.05, x_2 = 0.1$ of the interval $[0, 0.1]$, and $f(x) = e^{2x}$.

[i] Find the quadratic spline function s that interpolates f, and satisfies $s'(0) = f'(0)$.

[ii] Find the cubic spline function S that interpolates f, and satisfies $S'(0) = f'(0)$ and $S'(0.1) = f'(0.1)$.

[iii] Compute $s(0.02)$ and $S(0.02)$, using the functions which you have constructed in parts [i] and [ii], respectively. Then, compare both values to $e^{0.04} = 1.04081077$. Which interpolating function gives the least absolute error?

[100 marks]

4. (a) The Malthusian law of population growth predicts that under ideal conditions the population P at time t is

$$P(t) = P_0 e^{\lambda t}$$

where P_0 is the population at time $t = t_0$ and λ is a constant (called the population growth rate). Suppose that a population begins with 6 members and that observations are made to collect the following data:

t	3	5	6
P	15	25	40

Based on the method of least squares,

[i] state the population growth rate, λ,

[ii] what is the predicted population when $t = 8$?

3. (a)

x	$f(x)$	$f'(x)$
0.30	0.29552	0.95534
0.32	0.31457	0.94924
0.35	0.34290	0.93937

[i] *Guna nilai x dan aritmetik bundar lima digit untuk* $f(x)$ and $f'(x)$ *di atas untuk membina polinomial interpolasi Hermite,* $H_5(x)$.

[ii] *Seterusnya, guna polinomial yang telah dibina dalam bahagian [i] untuk menganggar* $f(0.34)$.

(b) *Andaikan petak* $x_0 = 0, x_1 = 0.05, x_2 = 0.1$ *pada selang* $[0, 0.1]$, *dan* $f(x) = e^{2x}$.

[i] *Cari fungsi splin kuadratik* s *yang menginterpolasi* f, *dan memenuhi* $s'(0) = f'(0)$.

[ii] *Cari fungsi splin kubik* S *yang menginterpolasi* f, *dan memenuhi* $S'(0) = f'(0)$ *dan* $S'(0.1) = f'(0.1)$.

[iii] *Menggunakan fungsi yang anda telah bina, masing-masingnya dalam bahagian [i] dan [ii], kira* $s(0.02)$ *dan* $S(0.02)$. *Seterusnya, bandingkan kedua-dua nilai tersebut dengan* $e^{0.04} = 1.04081077$. *Fungsi interpolasi manakah memberikan ralat mutlak yang terkecil?*

[100 markah]

4. (a) *Petua Malthus pertumbuhan jumlah penduduk meramalkan bahawa di bawah keadaan ideal, jumlah penduduk* P *pada masa* t *ialah*

$$P(t) = P_0 e^{\lambda t}$$

dengan P_0 *ialah jumlah penduduk pada masa* $t = t_0$ *dan* λ *ialah suatu pemalar (dinamai kadar pertumbuhan jumlah penduduk). Andaikan suatu jumlah penduduk bermula dengan 6 ahli dan cerapan dibuat untuk mengutip data berikut:*

t	3	5	6
P	15	25	40

Berdasarkan kaedah kuasa dua terkecil,
[i] nyatakan kadar pertumbuhan jumlah penduduk, λ,
[ii] berapakah jumlah penduduk yang dijangkakan apabila $t = 8$?

(b) For $x \in [-1,1]$, the Chebyshev polynomial is defined by $T_n(x) = \cos[n \arccos x]$, for each $n \geq 0$.

[i] Show that for any positive integers m and n with $m > n$,

$$T_m(x)T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)].$$

[ii] Write down the set of *monic* (polynomials with leading coefficient 1) Chebyshev polynomials $\{\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4\}$.

[iii] The function $f(x) = e^{-x}$ is approximated on the interval $[-1,1]$ by the n^{th} Maclaurin polynomial, $P_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{i!} x^i$. The fourth Maclaurin polynomial has truncation error $|R_4(x)| = \frac{|f^{(5)}(\xi(x))||x^5|}{120} \leq \frac{e}{120} \approx 0.0023.$ Use Chebyshev polynomials to obtain a lesser-degree polynomial approximation for $f(x)$ while keeping the error less than 0.05 on $[-1,1]$.

[100 marks]

(b) *Untuk* $x \in [-1,1]$, *polinomial Chebyshev ditakrifkan oleh* $T_n(x) = \cos[n \arccos x]$, *untuk setiap* $n \geq 0$.

[i] *Tunjukkan bahawa untuk sebarang integer positif* m *dan* n *dengan* $m > n$, $T_m(x)T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)]$.

[ii] *Tuliskan set polinomial Chebyshev monik* (*polinomial dengan pekali pelopor* 1) $\{\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4\}$.

[iii] *Fungsi* $f(x) = e^{-x}$ *dianggarkan atas selang* $[-1,1]$ *oleh polinomial Maclaurin ke-n,* $P_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{i!} x^i$. *Polinomial Maclaurin ke-4 mempunyai ralat pangkasan* $|R_4(x)| = \frac{|f^{(5)}(\xi(x))||x^5|}{120} \leq \frac{e}{120} \approx 0.0023$. *Guna polinomial Chebyshev untuk memperoleh hampiran polinomial darjah lebih rendah untuk* $f(x)$ *dengan mengekalkan ralat kurang dari* 0.05 *pada selang* $[-1, 1]$.

[100 markah]

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