UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Given that the distribution function $F_{X,Y}(x,y) = k(4x^2y^2 + 5xy^4),\ 0 < x < 1, 0 < y < 1$, find k and the corresponding probability density function and use it to calculate $P(0 < X < \frac{1}{2}, \frac{1}{2} < Y < 1)$.

[30 marks]

(b) Find the variance of Y if the moment generating function $M_Y(t) = e^{2t}/(1-t^2)$.

[20 marks]

(c) If $f_{X,Y}(x,y) = 2, x \geq 0, y \geq 0, x + y \leq 1$ show that the conditional probability density function of Y given $X = x$ is from a uniform distribution.

[20 marks]

(d) Let $f(x, y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, zero elsewhere, be the probability density function of X and Y. Then if $Z = X + Y$, find P ($Z \leq z$), for $0 < z < \infty$. What is the probability density function of Z?

[30 marks]

2. (a) Let X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Suppose $A = X/Y$ and $B = Y$. Find the joint probability density function (A, B) and hence the marginal probability density of A and B.

[30 marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the exponential probability density function $f_Y(y) = e^{-y}, y > 0$. What is the smallest n for which $P(Y_{\min} < 0.2) > 0.9$?

[30 marks]

1. (a) *Diberi bahawa fungsi taburan* $F_{X,Y}(x,y) = k(4x^2y^2 + 5xy^4),\ 0 < x < 1, 0 < y < 1$, *cari k dan fungsi ketumpatan kebarangkalian yang sepadan dan gunakannya untuk mengira* $P(0 < X < \frac{1}{2}, \frac{1}{2} < Y < 1)$.

[30 markah]

(b) *Cari varians bagi Y jika fungsi penjana momen* $M_Y(t) = e^{2t}/(1-t^2)$.

[20 markah]

(c) *Jika* $f_{X,Y}(x,y) = 2, x \geq 0, y \geq 0, x + y \leq 1$ *tunjukkan bahawa fungsi ketumpatan kebarangkalian bersyarat bagi Y diberi X = x adalah dari taburan seragam.*

[20 markah]

(d) *Biarkan* $f(x, y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, *sifar selainnya, adalah fungsi ketumpatan kebarangkalian bagi X dan Y. Kemudian jika Z = X + Y, cari* $P(Z \leq z)$, *untuk* $0 < z < \infty$. *Apakah fungsi ketumpatan kebarangkalian bagi Z?*

[30 markah]

2. (a) *Biarkan X dan Y mempunyai fungsi ketumpatan kebarangkalian tercantum*

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & di\ tempat\ lain \end{cases}$$

Andaikan A = X/Y dan B = Y. Cari fungsi ketumpatan kebarangkalian tercantum (A, B) dan seterusnya fungsi ketumpatan kebarangkalian sut bagi A dan B.

[30 markah]

(b) *Biarkan* $Y_1, Y_2, \ldots, Y_n$ *sebagai sampel rawak dari fungsi ketumpatan kebarangkalian eksponen* $f_Y(y) = e^{-y}, y > 0$. *Apakah nilai n yang terkecil bagi* $P(Y_{\min} < 0.2) > 0.9$?

[30 markah]

(c) If $X_1, X_2, \ldots, X_n$ are random samples from the distribution $N(\mu, 4\sigma^2)$, and $\overline{X}_m$ is defined as $\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i, \quad m \le n,$

Find the distribution of the following statistics:

(i) $m\overline{X}_m - n\overline{X}_n$

(ii) $(m-1)\overline{X}_m + (n-1)\overline{X}_n$

(iii) $\dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\displaystyle\sum_{i=1}^{m} \frac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

[40 marks]

3. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the Poisson distribution $p_X(k;\lambda) = \dfrac{\lambda^{2k} e^{-\lambda^2}}{k!},\ k = 0,1,2,\ldots$ Show that $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an efficient estimator for λ.

[30 marks]

(b) Show that $\hat{\sigma}^2 = \sum_{i=1}^{n} Y_i^2$ is sufficient for σ^2 if $Y_1, Y_2, \ldots, Y_n$ is a random sample from a normal pdf with $\mu = 0$.

[20 marks]

(c) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the pdf $f_Y(y;\theta) = \theta y^{\theta-1},\ 0 \le y \le 1.$ Show that $W = \prod_{i=1}^{n} Y_i$ is a sufficient estimator for θ. Is the maximum likelihood estimator of θ a function of W?

[30 marks]

(d) If $X_1, X_2, \ldots, X_n$ is a random sample from the Bernoulli distribution (p), $0 < p < 1$ and $\overline{X}_n$ is the sample mean, prove that $\overline{X}_n \xrightarrow{P} p$

[20 marks]

(c) Jika $X_1, X_2, \ldots, X_n$ adalah sampel rawak daripada taburan $N(\mu, 4\sigma^2)$, dan $\bar{X}_m$ adalah ditakrifkan sebagai $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i, \quad m \le n,$

Cari taburan bagi statistik berikut:

(i) $m\bar{X}_m - n\bar{X}_n$

(ii) $(m-1)\bar{X}_m + (n-1)\bar{X}_n$

(iii) $\dfrac{m(\bar{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\sum_{i=1}^{m} \dfrac{m(\bar{X}_m - \mu)^2}{4\sigma^2}$

[40 markah]

3. (a) Biarkan $X_1, X_2, \ldots, X_n$ sebagai sampel rawak bersaiz n dari taburan Poisson $p_X(k;\lambda) = \dfrac{\lambda^{2k} e^{-\lambda^2}}{k!}, \ k = 0,1,2,\ldots$. Tunjukkan bahawa $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$ adalah penganggar cekap bagi λ.

[30 markah]

(b) Tunjukkan bahawa $\hat{\sigma}^2 = \sum_{i=1}^{n} Y_i^2$ adalah cukup bagi σ^2 jika $Y_1, Y_2, \ldots, Y_n$ adalah sampel rawak daripada taburan normal yang mempunyai fungsi ketumpatan kebarangkalian dengan $\mu = 0$.

[20 markah]

(c) Biarkan $Y_1, Y_2, \ldots, Y_n$ sebagai sampel rawak bersaiz n daripada fungsi ketumpatan kebarangkalian $f_Y(y;\theta) = \theta y^{\theta-1}, \ 0 \le y \le 1.$ Tunjukkan bahawa $W = \prod_{i=1}^{n} Y_i$ adalah penganggar cukup bagi θ. Adakah penganggar kebolehjadian maksimum bagi θ suatu fungsi W?

[30 markah]

(d) Jika $X_1, X_2, \ldots, X_n$ sampel rawak daripada taburan Bernoulli (p), $0 < p < 1$ dan $\bar{X}_n$ ialah min sampel, buktikan bahawa $\bar{X}_n \xrightarrow{P} p$

[20 markah]

4. (a) Assume that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution, $N(\mu,4)$. Define $\bar{X}_n = \dfrac{\sum_{i=1}^{n} X_i}{n}$. Answer the following questions below:

(i) Is $\bar{X} - \mu$ a pivotal quantity? Explain.

(ii) Is $\dfrac{\bar{X}-\mu}{2/\sqrt{n}}$ a pivotal quantity? Explain.

(iii) Is $\dfrac{\bar{X}}{\mu}$ a pivotal quantity? Explain.

[20 marks]

(b) Assume that $X_1, X_2, \ldots, X_n$ is a random sample from the gamma distribution G(2,θ). Based on the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, construct

(i) approximate confidence interval for θ when n is big.

(ii) confidence interval for θ when n is small.

[40 marks]

(c) Assume that X is a single observation from a distribution with probability density function $f(x;\beta) = (1+\beta)x^{\beta} I_{(0,1)}(x)$, with $\beta > -1$.

(i) Find the most powerful test of size α to test $H_0 : \beta = 0$ versus $H_1 : \beta = 1$.

(ii) To test $H_0 : \beta \le 0$ versus $H_1 : \beta > 0$, the following test is used: Reject H_0 if $X \ge 3/4$. Find the power function and size for the test.

[40 marks]

4. (a) *Andaikan* $X_1, X_2, \ldots, X_n$ *sampel rawak daripada taburan normal, N(*μ*,4).* *Takrifkan* $\overline{X}_n = \dfrac{\sum_{i=1}^{n} X_i}{n}$*. Jawab setiap soalan di bawah:*

(i) *Adakah* $\overline{X} - \mu$ *suatu kuantiti pangsian? Jelaskan.*

(ii) *Adakah* $\dfrac{\overline{X} - \mu}{2/\sqrt{n}}$ *suatu kuantiti pangsian? Jelaskan.*

(iii) *Adakah* $\dfrac{\overline{X}}{\mu}$ *suatu kuantiti pangsian? Jelaskan.*

[20 markah]

(b) *Andaikan* $X_1, X_2, \ldots, X_n$ *suatu sampel rawak daripada taburan G(2,*θ*).* *Berdasarkan min sampel* $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *, terbitkan*

(i) *selang keyakinan hampiran bagi* θ *apabila n besar.*

(ii) *selang keyakinan bagi* θ *apabila n kecil.*

[40 markah]

(c) *Andaikan X suatu cerapan tunggal daripada taburan dengan fungsi ketumpatan* $f(x;\beta) = (1+\beta)x^{\beta} I_{(0,1)}(x)$*, dengan* $\beta > -1$*.*

(i) *Cari ujian paling berkuasa bersaiz* β *untuk menguji* $H_0 : \beta = 0$ *lawan* $H_1 : \beta = 1$*.*

(ii) *Untuk menguji* $H_0 : \beta \leq 0$ *lawan* $H_1 : \beta > 0$ *, ujian berikut digunakan: Tolak* H_0 *jika* $X \geq 3/4$*. Cari fungsi kuasa dan saiz bagi ujian tersebut.*

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\,\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}\, x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}\, x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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