UNIVERSITI SAINS MALAYSIA

First Semester Examination
2009/2010 Academic Session

November 2009

MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:*** *Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) A random variable X has a cumulative distribution function (c.d.f.) defined as

$$F(x) = \begin{matrix} 0 & , \ x < 0 \\ a + be^{-x} & , \ x \geq 0. \end{matrix}$$

(i) Find the values of a and b.

(ii) Determine $P\ 3 \leq X \leq 5$.

(iii) Construct and sketch the probability density function (p.d.f.) of X.

[35 marks]

(b) Suppose that earthquakes occur in a certain country at rate 1 per month. What is the probability that at least 4 earthquakes occur during the next 6 months?

[25 marks]

(c) After sitting for a trial examination, a student is given a grade of A, B, C, D or F based on their scores in the examination. From the previous record, 15% of the students were given a grade A; 40% were given a grade B; 25% were given a grade C, 15% were given a grade D and 5% were given a grade F. Furthermore, it was found that 0.5% of the students given a grade A eventually failed in a final examination, and the failure rate was 5% for students given a grade B, 15% for students given a grade C, 25% for students given a grade D and 60% for students given a grade F.

(i) If a student failed in the final examination, what is the probability that he/she received a grade D or F in the trial examination?

(ii) If a student did not fail in the final examination, what is the probability that he/she had received a grade A in the trial examination?

[40 marks]

2. (a) Suppose that X is a random variable with $E(X) = \mu$. Show that for every constant, $c \neq \mu$, $Var(X) \leq E\left[\ X - c \ ^2 \right]$.

[25 marks]

(b) The probability density function (p.d.f.) of X is given by

$$f(x) = \begin{cases} 1/8 \ , & x = 0 \text{ or } 2 \\ 3/4 \ , & x = 1 \\ 0 \ , & \text{elsewhere.} \end{cases}$$

(i) Determine the moment generating function (m.g.f.) of X.

(ii) From the m.g.f., find the mean and variance of X.

(iii) Show that for this random variable, equality holds in Chebyshev's inequality when $k = 1$.

[40 marks]

1. (a) *Pembolehubah rawak X mempunyai fungsi taburan longgokan (f.t.l.) yang ditakrif sebagai*

$$F(x)=\begin{matrix}0 & , & x<0\\ a+be^{-x} & , & x\geq 0.\end{matrix}$$

(i) Cari nilai a dan b.

(ii) Tentukan $P\ 3\leq X\leq 5$.

(iii) Bina dan lakar fungsi ketumpatan kebarangkalian (f.k.k.) bagi X.

[35 markah]

(b) *Andaikan gempa bumi berlaku di sebuah negara tertentu pada kadar 1 setiap bulan. Apakah kebarangkalian bahawa sekurang-kurangnya 4 gempa bumi berlaku dalam tempoh 6 bulan akan datang?*

[25 markah]

(c) *Selepas menduduki suatu peperiksaan percubaan, pelajar diberi gred A, B, C, D atau F berdasarkan pencapaian mereka dalam peperiksaan itu. Daripada rekod terdahulu, 15% pelajar diberi gred A; 40% diberi gred B; 25% diberi gred C, 15% diberi gred D and 5% diberi gred F. Juga, didapati bahawa 0.5% pelajar yang diberi gred A akhirnya gagal dalam peperiksaan akhir, dan kadar kegagalan adalah 5% bagi pelajar yang diberi gred B, 15% bagi pelajar yang diberi gred C, 25% bagi pelajar yang diberi gred D dan 60% bagi pelajar yang diberi gred F.*

(i) Jika pelajar gagal dalam peperiksaan akhir, apakah kebarangkalian yang beliau mendapat gred D atau F dalam peperiksaan percubaan?

(ii) Jika pelajar tidak gagal dalam peperiksaan akhir, apakah kebarangkalian beliau mendapat gred A dalam peperiksaan percubaan?

[40 markah]

2. (a) *Andaikan X adalah suatu pembolehubah rawak dengan* $E(X)=\mu$. *Tunjukkan bahawa untuk setiap pemalar,* $c\neq\mu, Var(X)\leq E\left[\ X-c\ ^2\right]$.

[25 markah]

(b) *Fungsi ketumpatan kebarangkalian (f.k.k.) bagi X diberi oleh*

$$f(x)=\begin{cases}1/8\ , & x=0\ atau\ 2\\ 3/4\ , & x=1\\ 0\ \ , & di\ tempat\ lain.\end{cases}$$

(i) Tentukan fungsi penjana momen (f.p.m.) bagi X.

(ii) Daripada f.p.m. tersebut, cari min dan varians bagi X.

(iii) Tunjukkan bahawa untuk pembolehubah rawak ini, persamaan dalam ketaksamaan Chebyshev dipenuhi bila k = 1.

[40 markah]

(c) A random variable R with p.d.f. given by

$$f(r)=\begin{cases}\frac{r}{\sigma^2}e^{-r^2/2\sigma^2}, & r>0\\ 0, & \text{elsewhere}\end{cases}$$

is called a Rayleigh random variable and is said to have the Rayleigh distribution with parameter σ^2. Determine and identify a p.d.f. of the random variable R^2.

[35 marks]

3. (a) Let X and Y have the joint p.d.f. given by

		y	
		-1	1
	-1	0.15	0.20
x	0	0.05	0.15
	1	0.20	0.25

(i) Determine $P\ |X-Y|\leq 1$ and $P\ X=0|Y=1$.

(ii) Find the covariance of X and Y.

(iii) If $Z=X^2+2Y$, determine the p.d.f. of Z.

[40 marks]

(b) Suppose that X and Y are random variables. The marginal p.d.f. of X is

$$f(x)=\begin{matrix}kx^2, & 0<x<1\\ 0, & \text{elsewhere.}\end{matrix}$$

Also, the conditional p.d.f. of Y given that $X = x$ is

$$h(y|x)=\begin{cases}\frac{3y^2}{x^3}, & 0<y<x\\ 0, & \text{elsewhere.}\end{cases}$$

Determine

(i) the value of k.

(ii) the marginal p.d.f. of Y. Are X and Y independent?

(iii) the conditional p.d.f. of X given that $Y = y$.

(iv) $E\ X^2|Y=y$.

[40 marks]

(c) Let $Z_1, Z_2, ..., Z_{24}$ be a random sample from the standard normal distribution $N(0,1)$. Let $W=Z_1^2+Z_2^2+...+Z_{24}^2$. Find $P(12.4<W<33.2)$.

[20 marks]

4. (a) Company A produces candies with vitamin C that have a label weight of 35.00 grams. Assume that the distribution of the weight of these candies is $N(35.50, 6.25)$.

(i) Let X denote the weight of a single candy selected at random from the production line. Find $P(X<30.60)$.

(c) *Pembolehubah rawak R dengan f.k.k. diberi oleh*

$$f(r)=\begin{cases}\dfrac{r}{\sigma^2}e^{-r^2/2\sigma^2}, & r>0\\ 0, & \text{di tempat lain}\end{cases}$$

dipanggil pembolehubah rawak Rayleigh dan dikatakan mempunyai taburan Rayleigh dengan parameter σ^2. *Dapatkan dan kenalpasti suatu f.k.k. bagi pembolehubah rawak* R^2.

[35 markah]

3. (a) *Biar X dan Y mempunyai f.k.k. tercantum diberi oleh*

		y	
		-1	1
x	-1	0.15	0.20
	0	0.05	0.15
	1	0.20	0.25

(i) *Tentukan dan* $P(|X-Y|\le 1)$ *dan* $P(X=0|Y=1)$.

(ii) *Cari kovarians bagi X dan Y.*

(iii) *Jika* $Z=X^2+2Y$, *dapatkan f.k.k. bagi Z.*

[40 markah]

(b) *Andaikan X dan Y adalah dua pembolehubah rawak. F.k.k. sut bagi X adalah*

$$f(x)=\begin{cases}kx^2, & 0<x<1\\ 0, & \text{di tempat lain.}\end{cases}$$

Juga, f.k.k. bersyarat bagi Y diberi X = x adalah

$$h(y|x)=\begin{cases}\dfrac{3y^2}{x^3}, & 0<y<x\\ 0, & \text{di tempat lain.}\end{cases}$$

Tentukan

(i) *nilai k.*

(ii) *f.k.k. sut bagi Y. Adakah X dan Y tidak bersandar?*

(iii) *f.k.k. bersyarat bagi X diberi Y = y.*

(iv) $E(X^2|Y=y)$.

[40 markah]

(c) *Biar* $Z_1, Z_2, ..., Z_{24}$ *menjadi suatu sampel rawak daripada taburan normal piawai N(0,1). Biar* $W=Z_1^2+Z_2^2+...+Z_{24}^2$. *Cari* $P(12.4<W<33.2)$.

[20 markah]

4. (a) *Syarikat A menghasilkan gula-gula bervitamin C yang mempunyai label berat 35.00 gram. Andaikan taburan berat gula-gula ini adalah N(35.50, 6.25).*

(i) *Biar X sebagai berat satu gula-gula yang dipilih secara rawak daripada baris pengeluaran. Cari* $P(X<30.60)$.

(ii) Suppose that 120 candies are selected at random and weighed. Let Y equal the number of these candies that weigh less than 30.60 grams. Approximate $P(Y < 3)$.

(iii) Let $\bar{X}$ equal the sample mean of the weight of these 120 selected candies. Find $P\ 34.82 < \bar{X} < 35.84$.

[30 marks]

(b) Suppose that $\bar{X}$ and S^2 are the mean and variance of a random sample of size 16 from the normal distribution $N(10,64)$.

(i) Find $P(6 < \bar{X} < 14)$.

(ii) Determine $P\ 19.63 < S^2 < 130.47$.

[20 marks]

(c) Suppose that X and Y are two independent random variables, each with a $\chi^2(2)$ distribution. Show that the random variables $U = \frac{X}{Y}$ and $V = X + Y$ are independent.

[50 marks]

(ii) *Andaikan 120 gula-gula dipilih secara rawak dan ditimbang. Biar Y sama dengan bilangan gula-gula ini yang mempunyai berat kurang daripada 30.60 gram. Anggarkan* $P(Y < 3)$.

(iii) *Biar* $\bar{X}$ *sama dengan min sampel berat 120 gula-gula yang dipilih ini. Cari* $P\ 34.82 < \bar{X} < 35.84$.

[30 markah]

(b) *Andaikan bahawa* $\bar{X}$ dan S^2 *adalah min dan varians suatu sampel rawak bersaiz 16 daripada taburan normal N(10,64).*

(i) *Cari* $P(4 < \bar{X} < 16)$.

(ii) *Tentukan* $P\ 19.63 < S^2 < 130.47$.

[20 markah]

(c) *Andaikan X dan Y adalah dua pembolehubah rawak tak bersandar, setiap satu dengan taburan* $\chi^2(2)$. *Tunjukkan pembolehubah-pembolehubah rawak* $U = \frac{X}{Y}$ *dan V = X + Y adalah tak bersandar.*

[50 markah]

Appendix/Lampiran

	Probability Density Function (P.D.F.)
Bernoulli	$p^x(1-p)^{1-x}, \quad x=0,1 \quad , \quad 0<p<1$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}, \quad x=0,1,...,n \quad , \quad 0<p<1$
Hypergeometric	$\frac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}, \quad x=0,1,... \quad , r \le n \text{ or } x=1,2,... \quad , n_1 \le r$
Geometric	$(1-p)^{x-1}p, \quad x=1,2,...$
Negative Binomial	$\binom{x-1}{r-1}p^r(1-p)^{x-r}, \quad x=r,r+1,...$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}, \quad x=0,1,2,... \ , \quad \lambda>0$
Uniform	$\frac{1}{\beta-\alpha}, \quad \alpha<x<\beta$
Normal	$\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty<x<\infty$
Exponential	$\lambda e^{-\lambda x}, \quad x \ge 0$
Gamma	$\frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}, \quad x \ge 0, \quad \lambda>0, \quad \alpha>0$
Chi-square	$\frac{1}{\Gamma(r/2)\,2^{r/2}}x^{r/2-1}e^{-x/2}, \quad x \ge 0$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0<x<1, \quad \alpha>0, \quad \beta>0$

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