UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2009/2010

Jun 2010

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) If X has a normal distribution, $N(0,\sigma^2)$, with density function given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2}\right)I_{(-\infty,\infty)}(x),$$

find the density for $Y=|X|$.

[30 marks]

(b) Show that if X and Y are independent positive continuous random variables, then the density function for the random variable $Z=\frac{X}{Y}$ is

$$g(z)=\int_0^\infty y f_X(zy) f_Y(y)dy.$$

[30 marks]

(c) The joint density function X_1 and X_2 is

$$f(x_1,x_2)=\frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2},\quad -\infty<x_1,x_2<\infty.$$

Let $Y_1=X_1+X_2$ and $Y_2=X_1-X_2$. Find the joint density function for Y_1 and Y_2.

[40 marks]

2. (a) Let $Y_1<Y_2<Y_3<Y_4$ be the order statistics of a random sample of size 4 from the distribution having probability density function

$$f(x)=e^{-x}, 0<x<\infty, \text{ zero elsewhere.}$$

Find $P(3\leq Y_4)$.

[30 marks]

(b) Differentiate the moment-generating function for the geometric random variable and show that $E(X)=(1-p)/p$ and $\mathrm{Var}(X)=(1-p)/p^2$.

[30 marks]

(c) (i) Let U has an F distribution with parameters r and s. Using the related theorem, prove that $\frac{1}{U}$ has an F distribution with parameters s and r.

(ii) Let $T=\frac{W}{\sqrt{V/r}}$, where the independent variables W and V follow the standard normal distribution and the chi-square distribution with r degrees of freedom, respectively. Using the related theorem, show that T^2 has an F distribution with parameters 1 and r.

[40 marks]

1. (a) *Jika X mempunyai taburan normal,* $N(0,\sigma^2)$, *dengan fungsi ketumpatan yang diberikan oleh*

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,\text{eks}\left(-\frac{x^2}{2\sigma^2}\right)I_{(-\infty,\infty)}(x),$$

cari fungsi ketumpatan untuk $Y=|X|$.

[30 markah]

(b) *Tunjukkan bahawa jika X dan Y pembolehubah rawak selanjar positif yang tak bersandar, maka fungsi ketumpatan pembolehubah rawak* $Z=\dfrac{X}{Y}$ *ialah*

$$g(z)=\int_0^\infty y f_X(zy)f_Y(y)\,dy.$$

[30 markah]

(c) *Fungsi ketumpatan tercantum* X_1 *dan* X_2 *ialah*

$$f(x_1,x_2)=\frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2},\quad -\infty<x_1,x_2<\infty.$$

Biarkan $Y_1=X_1+X_2$ *dan* $Y_2=X_1-X_2$. *Cari fungsi ketumpatan tercantum* Y_1 *dan* Y_2.

[40 markah]

2. (a) *Biarkan* $Y_1<Y_2<Y_3<Y_4$ *sebagai statistik tertib bagi suatu sampel rawak saiz 4 daripada suatu taburan dengan fungsi ketumpatan kebarangkalian*

$$f(x)=e^{-x},0<x<\infty,\ \textit{sifar di tempat lain.}$$

Cari $P(3\le Y_4)$.

[30 markah]

(b) *Bezakan fungsi penjana momen bagi pembolehubah rawak geometri dan tunjukkan bahawa* $E(X)=(1-p)/p$ *dan* $Var(X)=(1-p)/p^2$.

[30 markah]

(c) (i) *Biarkan U mempunyai taburan F dengan parameter r dan s. Dengan menggunakan teorem yang berkaitan, buktikan bahawa* $\dfrac{1}{U}$ *mempunyai taburan F dengan parameter s dan r.*

(ii) *Biarkan* $T=\dfrac{W}{\sqrt{V/r}}$, *yang mana pembolehubah tak bersandar W dan V, masing-masing mengikuti taburan normal piawai dan taburan khi kuasa dua dengan r darjah kebebasan. Dengan menggunakan teorem yang berkaitan, tunjukkan bahawa* T^2 *mempunyai taburan F dengan parameter 1 dan r.*

[40 markah]

3. (a) Let $X_1, X_2, ..., X_n$ represent a random sample from the $N(\mu,1)$ distribution. Show that the minimum variance of an arbitrary unbiased estimator μ^2 from the Cramer-Rao's inequality is $\dfrac{4\mu^2}{n}$.

[30 marks]

(b) Assume that $X_1, X_2, ..., X_n$ represent a random sample from a common distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta(1-\theta)^{-1} x^{(2\theta-1)/(1-\theta)}, & 0 < x \le 1,\ \tfrac{1}{2} < \theta < 1 \\ 0 & , \text{elsewhere} \end{cases}.$$

Find the maximum likelihood estimator for θ.

[30 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from the $G(5,\theta)$ distribution. By using the sample mean $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$, derive the

(i) approximate $100\gamma\%$ $(0<\gamma<1)$ confidence interval for θ when n is large.

(ii) $100\gamma\%$ $(0<\gamma<1)$ confidence interval for θ when n is small.

[40 marks]

3. (a) *Biarkan* $X_1, X_2, ..., X_n$ *mewakili suatu sampel rawak daripada taburan* $N(\mu,1)$. *Tunjukkan bahawa varians minimum sebarang penganggar saksama untuk* μ^2 *daripada ketaksamaan Cramer-Rao ialah* $\dfrac{4\mu^2}{n}$.

[30 markah]

(b) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *mewakili sampel rawak daripada taburan sepunya dengan fungsi ketumpatan kebarangkalian*

$$f(x;\theta) = \begin{cases} \theta(1-\theta)^{-1} x^{(2\theta-1)/(1-\theta)}, & 0 < x \le 1, \ \tfrac{1}{2} < \theta < 1 \\ 0 & , \textit{di tempat lain} \end{cases}.$$

Cari penganggar kebolehjadian maksimum bagi θ.

[30 markah]

(c) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *adalah suatu sampel rawak daripada taburan* $G(5,\theta)$. *Dengan menggunakan min sampel* $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$, *terbitkan*

(i) *selang keyakinan hampiran* 100γ % $(0 < \gamma < 1)$ *bagi* θ *apabila n besar.*

(ii) *selang keyakinan* 100γ % $(0 < \gamma < 1)$ *bagi* θ *apabila n kecil.*

[40 markah]

4. (a) Assume that $X_1, X_2, \ldots, X_n$ is a random sample from the normal distribution with mean, μ and an unknown variance. Then

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$

is a pivotal quantity. Find the shortest 100γ percent confidence interval for μ.

[30 marks]

(b) Assume that the random variable X has the density function

$$f(x;\beta) = \begin{cases} \beta e^{-\beta x}, & 0 < x < \infty \\ 0, & \text{elsewher} \end{cases}.$$

A random sample of size two, X_1 and X_2, is taken to test $H_0 : \beta = \frac{1}{2}$ versus $H_1 : \beta = \frac{1}{4}$. The test used is based on the critical region $C = \{(x_1, x_2) : x_1 + x_2 \geq 9.5\}$. Find the power of the test used.

[30 marks]

(c) Let $X_1, X_2, \ldots, X_n$ represent a random sample from the Poisson distribution with density function

$$f(x;\lambda) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\ldots\}}(x).$$

Find the uniformly most powerful (UMP) test of size α = 0.1 to test $H_0 : \lambda = 2$ versus $H_1 : \lambda > 2$. Assume that the sample size is n = 30.

[40 marks]

4. (a) *Andaikan* $X_1, X_2, ..., X_n$ *sampel rawak daripada taburan normal dengan min,* μ *dan varians yang tidak diketahui. Maka*

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t\,(n-1)$$

ialah suatu kuantiti pangsian. Cari selang keyakinan 100γ *peratus tersingkat bagi* μ.

[30 markah]

(b) *Andaikan pembolehubah rawak X mempunyai fungsi ketumpatan*

$$f(x;\beta) = \begin{cases} \beta e^{-\beta x}, & 0 < x < \infty \\ 0, & \text{dtl} \end{cases}$$

Sampel rawak saiz dua, $X_1\, dan X_2$, *diambil bagi menguji* $H_0 : \beta = \frac{1}{2}$ *lawan* $H_1 : \beta = \frac{1}{4}$. *Ujian yang digunakan adalah berdasarkan rantau genting C =* $\{(x_1, x_2) : x_1 + x_2 \geq 9.5\}$. *Cari kuasa bagi ujian yang digunakan.*

[30 markah]

(c) *Biarkan* $X_1, X_2, ..., X_n$ *menandai sampel rawak daripada taburan Poisson dengan fungsi ketumpatan*

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} I_{\{0,1,...\}}(x)$$

Cari ujian paling berkuasa secara seragam (UPBS) bersaiz $\alpha = 0.1$ *bagi menguji* $H_0 : \lambda = 2$ *lawan* $H_1 : \lambda > 2$. *Andaikan bahawa saiz sampel ialah n = 30.*

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\,I_{\{1,2,\ldots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\,I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\,I_{\{0,1,\ldots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\,I_{\{0,1,\ldots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\,I_{\{0,1,\ldots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\,I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,exp\{-(x-\mu)^2/2\sigma^2\}\,I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\,I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}\,I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}e^{-x/2}x^{(r/2)-1}\,I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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