UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2009/2010

Jun 2010

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) (i) If $\begin{bmatrix} 2 & y & y+1 \\ y-1 & z & x+1 \end{bmatrix} = \begin{bmatrix} 2 & y & 3 \\ z & z & y+2 \end{bmatrix}$, find x, y and z.

(ii) A square matrix is called *upper triangular* if all of the entries below the main diagonal are zero. Prove that the product of two upper triangular $n \times n$ matrices is upper triangular.

(b) Given the following matrices

$$A = \begin{bmatrix} 11 & -2 & 1 \\ 5 & 0 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 4 & 0 \\ 2 & 2 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}.$$

(i) If possible, evaluate $2CA - B$.

(ii) If possible, evaluate $B(B^{-1}CA + I_3) - 3CA$.

(iii) Is AC singular or non-singular? Justify.

(c) (i) Given that $A \in M_{n\times n}(\square)$. Prove that if $B = A + A^T$ and $C = A - A^T$ then B is symmetric and C is skew-symmetric.
[Recall that A^T is the transpose of A]

(ii) Let P and Q be $n \times n$ matrices with the property that $xP = xQ$ for all $x \in \square^n$. Show that $P = Q$.

(d) The augmented matrix of a system of equations in variables x_1 , x_2 , x_3 , x_4 , x_5 is reduced to the following form:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 1 & 2 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(i) For what (if any) value(s) of a, b is the system inconsistent?

(ii) For what (if any) value(s) of a, b does the system have a unique solution?

(iii) Solve the system in the case $a = b = 0$.

(iv) Solve the system in the case $a = b = 1$.

[100 marks]

1. (a) (i) *Jika* $\begin{bmatrix} 2 & y & y+1 \\ y-1 & z & x+1 \end{bmatrix} = \begin{bmatrix} 2 & y & 3 \\ z & z & y+2 \end{bmatrix}$, *cari x, y and z.*

(ii) *Suatu matriks segiempat sama dipanggil segitiga atas jika semua pemasukan di bawah pepenjuru adalah sifar. Buktikan bahawa hasildarab dua matriks segitiga atas* $n \times n$ *adalah segitiga atas.*

(b) *Diberi matriks berikut*

$$A = \begin{bmatrix} 11 & -2 & 1 \\ 5 & 0 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 4 & 0 \\ 2 & 2 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}.$$

(i) *Jika mungkin, nilaikan* $2CA - B$.

(ii) *Jika mungkin, nilaikan* $B(B^{-1}CA + I_3) - 3CA$.

(iii) *Adakah AC singular atau tak singular? Beri justifikasi.*

(c) (i) *Diberi* $A \in M_{n\times n}(\square)$. *Buktikan bahawa jika* $B = A + A^T$ *dan* $C = A - A^T$ *maka B simetri dan C simetri pencong.*
[Ingat bahawa A^T *ialah transposisi A]*

(ii) *Biar P dan Q sebagai matriks* $n \times n$ *dengan* $xP = xQ$ *untuk semua* $x \in \square^n$. *Tunjukkan bahawa* $P = Q$.

(d) *Matriks imbuhan bagi system persamaan linear dengan pembolehubah* x_1, x_2, x_3, x_4, x_5 *diturunkan kepada:*

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 1 & 2 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(i) *Apakah nilai a, b (jika ada) yang mana sistem tersebut menjadi tak konsisten.*

(ii) *Apakah nilai a, b (jika ada) yang mana sistem tersebut mempunyai penyelesaian unik.*

(iii) *Selesaikan system tersebut bagi kes a = b = 0.*

(iv) *Selesaikan system tersebut bagi kes a = b = 1.*

[100 markah]

2. (a) Let $A=\begin{bmatrix}1 & 1 & 0\\ 2 & 1 & 1\\ 3 & 2 & 2\end{bmatrix}$ and $b=(2,-1,0)$.

(i) Find A^{-1} using the Gauss-Jordan procedure.

(ii) Use your answer in (i) to solve for v where $vA=b$.

(b) (i) Let $S=\{v_1,v_2,v_3\}$ where $v_1=(1,-1,2)$, $v_2=(2,-1,2)$ and $v_3=(-4,1,-2)$. Find a basis for the linear span of S, $\mathcal{L}(S)$.

(ii) Find a basis for $\mathbb{R}^3$ containing the vector $(2,2,-1)$.

[Consider the vectors in the standard basis of $\mathbb{R}^3$]

(c) For each of the following, determine whether or not S is a subspace of the indicated vector space. Prove your claim.

(i) $S=\{ax^3+bx^2+cx+d \mid ad>0\}$ in $P_3(\mathbb{R})$.

(ii) $S=\{(x_1,x_2,x_3,x_4) \mid x_1=x_2+2 \text{ and } x_3=3x_4\}$ in $\mathbb{R}^4$.

(iii) $S=\left\{\begin{bmatrix}a & 0\\ b & 0\end{bmatrix} \middle| \, a,b \text{ are real numbers}\right\}$ in $M_{2\times 2}(\mathbb{R})$.

(d) Given

$$A=\begin{bmatrix}1 & 0 & -2 & 1 & 3\\ -1 & 1 & 5 & -1 & -3\\ 0 & 2 & 6 & 0 & 1\\ 1 & 1 & 1 & 1 & 4\end{bmatrix} \text{ where } \mathrm{RREF}(A)=\begin{bmatrix}1 & 0 & -2 & 1 & 0\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

where RREF(A) is the reduced row echelon form of A.

(i) Determine the column space of A, $C(A)$ and its column rank, $c(A)$.

(ii) Determine the row space of A, $R(A)$ and its row rank, $r(A)$.

(iii) Given that the solution for the system $Av=\underline{0}$ is the null space of A, $N(A)$. Find $N(A)$ and its dimension, $nullity(A)$.

(iv) Deduce the rank of A, $\rho(A)$ an verify that $\rho(A)+nullity(A)=5$.

[100 marks]

2. (a) *Biar* $A=\begin{bmatrix}1&1&0\\2&1&1\\3&2&2\end{bmatrix}$ *dan* $b=(2,-1,0)$.

(i) *Cari* A^{-1} *menggunakan prosedur Gauss-Jordan.*

(ii) *Guna jawapan anda dalam (i) untuk mendapatkan v bagi persamaan* $vA=b$.

(b) (i) *Biar* $S=\{v_1,v_2,v_3\}$ *supaya* $v_1=(1,-1,2)$, $v_2=(2,-1,2)$ *dan* $v_3=(-4,1,-2)$. *Cari asas untuk rentangan linear T,* $\mathscr{L}(T)$.

(ii) *Cari asas bagi* $\mathbb{R}^3$ *yang mengandungi vektor* $(2,2,-1)$.

[Pertimbangkan vektor-vektor dalam asas piawai $\mathbb{R}^3$*]*

(c) *Untuk setiap daripada berikut, tentukan sama ada S adalah subruang bagi ruang vektor yang tertunjuk. Buktikan tuntutan anda.*

(i) $S=\{ax^3+bx^2+cx+d \mid ad>0\}$ dalam $P_3(\mathbb{R})$.

(ii) $S=\{(x_1,x_2,x_3,x_4) \mid x_1=x_2+2 \text{ dan } x_3=3x_4\}$ dalam $\mathbb{R}^4$.

(iii) $S=\left\{\begin{bmatrix}a&0\\b&0\end{bmatrix} \middle| \, a,b \text{ adalah nombor nyata}\right\}$ dalam $M_{2\times 2}(\mathbb{R})$.

(d) *Diberi*

$$A=\begin{bmatrix}1&0&-2&1&3\\-1&1&5&-1&-3\\0&2&6&0&1\\1&1&1&1&4\end{bmatrix} \text{ dengan } \mathrm{RREF}(A)=\begin{bmatrix}1&0&-2&1&0\\0&1&3&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

yang mana RREF(A) ialah bentuk baris terturun bagi A.

(i) *Tentukan ruang lajur bagi A, C(A) dan pangkat lajurnya, c(A).*

(ii) *Tentukan ruang baris bagi A, R(A) dan pangkat barisnya, r(A).*

(iii) *Diberi bahawa penyelesaian bagi sistem* $Av=\underline{0}$ *merupakan ruang nol dari A. Cari* $N(A)$ *dan dimensinya,* $nullity(A)$.

(iv) *Buat kesimpulan tentang pangkat A,* $\rho(A)$ *dan tentusahkan yang*

$$\rho(A)+nullity(A)=5.$$

[100 markah]

3. (a) Consider some linearly independent vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$ and a vector v in $\mathbb{R}^n$ which is not a linear combination of $v_1, v_2, \ldots, v_m$. Are the vectors $v_1, v_2, \ldots, v_m, v$ necessarily linearly independent? Justify your answer.

(b) Find an example for each of the following or explain why no such example can be found.

(i) A set of four vectors in $M_{2\times 2}\ \mathbb{R}$ that is linearly independent.

(ii) A basis for $P_3\ \mathbb{R}$ containing three vectors.

(iii) A set of three vectors in $\mathbb{R}^5$ that spans $\mathbb{R}^5$.

(iv) A set of three vectors in $\mathbb{R}^3$ that is not a basis for $\mathbb{R}^3$.

(c) Given the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$x, y, z\ T = \ x + y, y + z, 0\ .$$

(i) Show that T is a linear transformation.

(ii) Find the kernel of T, Ker T.

(iii) Find a basis for Ker T.

(iv) Find a basis for the image of T, Im T.

(v) Is T one-to-one? Explain your answer.

(vi) Is T onto? Explain your answer.

(d) Given $W = \mathscr{L}\ \ 1,1,0\ ,\ 0,1,2\ \ $.

(i) Using the Gram-Schmidt process, find an orthonormal basis of W.

(ii) Find the orthogonal complement of W, $W^\perp$.

(iii) Show that $\mathbb{R}^3 = W \oplus W^\perp$.

[Hint: Show that $\mathbb{R}^3 = W + W^\perp$ and $W \cap W^\perp = \varnothing$]

[100 marks]

3. (a) *Pertimbangkan beberapa vektor* $v_1, v_2, \ldots, v_m$ *dalam* $\mathbb{R}^n$ *dan suatu vektor v dalam* $\mathbb{R}^n$ *yang bukan gabungan linear* $v_1, v_2, \ldots, v_m$. *Adakah vektor-vektor* $v_1, v_2, \ldots, v_m, v$ *semestinya tak bersandar linear? Beri justifikasi anda.*

(b) *Cari satu contoh untuk setiap yang berikut atau terangkan mengapa tiada contoh yang boleh dicari.*

(i) *Suatu set empat vektor dalam* $M_{2\times 2}(\mathbb{R})$ *yang tak bersandar linear.*

(ii) *Suatu asas bagi* $P_3(\mathbb{R})$ *yang mengandungi tiga vektor.*

(iii) *Suatu set tiga vektor dalam* $\mathbb{R}^5$ *yang merentang* $\mathbb{R}^5$.

(iv) *Suatu set tiga vektor dalam* $\mathbb{R}^3$ *yang bukan asas bagi* $\mathbb{R}^3$.

(c) *Diberi fungsi* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *yang ditakrifkan dengan*

$$(x, y, z)T = (x+y,\ y+z,\ 0).$$

(i) *Tunjukkan bahawa T ialah suatu transformasi linear.*

(ii) *Cari inti bagi T, Ker T.*

(iii) *Cari asas bagi Ker T.*

(iv) *Cari asas untuk imej bagi T, Im T.*

(v) *Adakah T satu-ke-satu? Terangkan jawapan anda.*

(vi) *Adakah T keseluruhan? Terangkan jawapan anda.*

(d) *Diberi* $W = \mathscr{L}\{(1,1,0), (0,1,2)\}$.

(i) *Menggunakan proses Gram-Schmidt, cari asas ortonormal bagi W.*

(ii) *Cari pelengkap berortogon bagi W,* $W^{\perp}$.

(iii) *Tunjukkan bahawa* $\mathbb{R}^4 = W \oplus W^{\perp}$.

[Petunjuk: Tunjukkan bahawa $\mathbb{R}^3 = W + W^{\perp}$ *dan* $W \cap W^{\perp} = \varnothing$*]*

[100 markah]

4. (a) Find the quadratic function that best fits the points $(-1,8)$, $(0,8)$, $(1,4)$ and $(2,16)$.

(b) Given that $T:\mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation where α and β are bases for $\mathbb{R}^3$ such that $\beta = \{(1,2),(1,-1)\}$. Suppose

$$[(x,y,z)]_\alpha = (y-z, z, x+y-z) \quad \text{for all} \quad (x,y,z) \in \mathbb{R}^3 \text{ and } [T]_{\alpha,\beta} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

find the definition $(x,y,z)T$ for T.

(c) Given

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}.$$

Find matrices P and D such that $PAP^{-1} = D$.
[Hint: $\det(A-\lambda I) = -(\lambda-3)(\lambda^2-9\lambda+18)$]

(d) Use the Cayley-Hamilton theorem to find B^{-1} if $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

[100 marks]

4. (a) *Cari fungsi quadratik terbaik yang memadankan titik-titik* $-1,8$, $0,8$, $1,4$ *dan* $2,16$.

(b) *Diberi* $T:\mathbb{R}^3 \to \mathbb{R}^2$ *ialah suatu transformasi linear yang mana* α *dan* β *adalah asas-asas bagi* $\mathbb{R}^3$ *sedemikian hingga* $\beta = \ 1,2 \ , \ 1,-1$. *Andai* $x,y,z \ _{\alpha} = \ y-z, z, x+y-z$ *untuk semua* $x,y,z \in \mathbb{R}^3$ *dan*

$$T_{\alpha,\beta} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

cari takrif $x,y,z \ T$ *bagi T.*

(c) *Diberi*

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}.$$

Cari matriks-matriks P dan D sedemikian hingga $PAP^{-1} = D$.
[Petunjuk: $\det \ A-\lambda I \ = -(\lambda-3)(\lambda^2-9\lambda+18)$ *]*

(d) *Guna teorem Cayley-Hamilton untuk mencari* B^{-1} *jika* $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

[100 markah]

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