ON THE NUMBER OF FAMILIES OF BRANCHING PROCESSES WITH IMMIGRATION WITH FAMILY SIZES WITHIN RANDOM INTERVAL

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ABSTRACT

We consider the number of families in Bienayme-Galton-Watson branching processes whose size is within a random interval. We obtain the limit theorems for the number of families in the $(n+1)$ st generation whose family size within the random interval for non-critical processes with immigration.

1. INTRODUCTION

Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual untill by the end of its lifetime, have produce k new offsprings with probability $p_k, k \geq 0$ independently of the number produced by any other individual. The number of individual at time t, $X(t), t \in N_0 = \{0,1,2,...\}$ is called the Bienayme-Galtan-Watson (BGW) process.

Let $X_{ki}, k \in N_0, i = N = \{0,1,2,...\}$ be independent and identically distributed random variables, taking the values in the set N_0. We define the process $X(t),\ t \in N,$ by the relation

$$X(0) = 1, \qquad X(t+1) = \sum_{i=1}^{X(t)} X_{ti}$$

Here, X_{ti} denotes the number of descendants of the i th individual existing at time t.

Assume at the time of birth of the t^{th} generation, that is, at time, there is an immigration of Y_t individuals into the population. Then the BGW process with immigration (BGWI) is defined by a sequence of a random variable $Z(t)$ which are determined by the relation

$$Z(t+1)=\sum_{i=1}^{z(t)} X_{ti}+Y_{t+1}$$

where the $Y_1,Y_2,...$ are independent and identically distributed with common generation function

$$h(u)=tu^{Y_1}=\sum_{k=0}^{\infty} P(Y_i=k)u^k$$

and these $Y_i's$ are independent of the random variable $\{X_{ti}\}$ which have common probability generating function $f(u)=\sum_{k=1}^{\infty} u^k p_X(t)$.Thus the probability generating function of $Z(t+1)$ is

$$g_{t+1}(u)=h(u)g_t\left[f(u)\right] \qquad (1)$$

Now let us consider the independent BGWZ defined by

$$S_{Z_{i(t)}}^{W}=\sum_{i=1}^{Z(t)} \varepsilon_i(t)$$

where

$$\varepsilon_i(t)=\begin{cases}1 & \text{if } X_{ti}\in(W_L,W_U) \\ 0 & \text{if } X_{ti}\notin(W_L,W_U)\end{cases} \qquad i=1,2,...$$

for a random interval $W=(W_L,W_u)$. We assume $X_{ti}'s$ are iid variables with a common cdf and independent from this random interval (W_L,W_U) and $Z(t)$.

If we let $Z(t)$ to be the number of families including the immigrating one from time t, then the random variable $S_{Z(t)}^{W}$ can now be interpreted as the number of families (including families with immigrating parents) living in the $(t+1)$ st generation who have family size within the random interval (X_L, X_U).

2. CRITICAL PROCESSES

First we consider the critical process. It is known that (see Jagers [1975]) that if

$$f'(t)=1, \quad f''(t)=\sigma^2<\infty \text{ and } \quad 0<h'(1)=\mu<\infty \qquad (2)$$

then $2Z_n(t)/\sigma^2 t$ converges in the distribution to a random variable with the gamma density function

$$w(x)=\frac{1}{\Gamma\left(\frac{2u}{\sigma^2}\right)}x^{\left(\frac{2u}{\sigma^2}\right)-1}e^{-x},\quad u\in(0,\infty) \tag{3}$$

Theorem 1: if (2) is satisfied, then

$$\lim_{t\to\infty}P\left(\frac{2S^W_{Z_{(t)}}}{\sigma^2 t}\le z\right)=P\left(Z\Delta(x)\le Z\right) \tag{4}$$

where Z is a random variable with density function (3) and

$$\Delta(x)=G(W_u)-G(W_L) \tag{5}$$

Proof: It is clear that for fixed $Z(t)$ and W $S^W_{Z(t)}$ is a Binomial random variable. Thus,

$$\begin{aligned}P\left(S^W_{Z(t)}=j\right)&=E\left[P\left(S^W_{Z(t)}=j\,\middle|\,Z(t),W\right)\right]\\&=E\left\{\binom{Z(t)}{j}\left(\Delta(x)\right)^j\left[1-\Delta(x)\right]^{Z(t)-j}\right\}.\end{aligned} \tag{6}$$

We find the Laplace transform of $Q(t)S^W_{Z(t)}$:

$$\begin{aligned}E\left(e^{-\lambda Q(t)S^W_{Z(t)}}\right)&=E\left\{\sum_{k=0}^{\infty}\sum_{j=0}^{k}e^{-\lambda jQ(t)}\binom{k}{j}\left(\Delta(x)\right)^j\right\}\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}e^{-\lambda jQ(t)}\binom{k}{j}\left[\Delta(x)\right]^j\left[1-\Delta(x)\right]^{k-j}P\left(Z(t)=k\,\middle|\,B(t)\right)\\&=E\left\{\sum_{k=0}^{\infty}\left[1-\Delta(x)+\Delta(x)e^{-\lambda Q(t)}\right]^{\frac{1}{Q(t)}kQ(t)}P\left[Z(t)Q(t)=kQ(t)\,\middle|\,B(t)\right]\right\}\\&\to E\left\{\int_0^{\infty}e^{-\lambda\Delta(x)y}dP(V\le g)\right\}\\&=E\left\{e^{-\lambda z\Delta(x)}\right\}\end{aligned}$$

by the expression $\lim_{w\to\infty}\left(1-a+ae^{b/w}\right)^w=e^{ab}$ and thus

$$x\log\left(1-a+ae^{b/w}\right)^w=x\log\left(1+\frac{ab}{w}+0\left(\frac{1}{w}\right)\right)=w\left(ab/w+0\left(\frac{1}{w}\right)\right)\to ab$$

Thus, we have equation (4) by taking into account the limit theorem in (3).

3. NON-CRITICAL PROCESSES

In the supercritical case, when the mean number of offspring $1 < m < \infty$, then there exists a sequence of consists C_t such that $\{Z(t)/C_t\}$ converges with probability 1 to a random variable V (see Jagers [1975]). In this case, if $E\left(\log I_1\right) < \infty$, then $P\left(V < \infty\right) = 1$ and V has an absolutely continuous distribution on $\left(0, \infty\right)$. If $E\left(\log I_1\right) = \infty$, then $P\left(V < \infty\right) = 0$

Theorem 2: Assume $1 < m < \infty$, then

$$\lim_{t \to \infty} P\left(S_{Z(t)}^{W} \big/ C_t \leq x\right) = P\left(V \Delta\left(x\right) \leq x\right)$$

where V is as mention in the limit theorem above.

Proof: The proof of this theorem is similar to those for Theorem 1, except we take into account the limit theorem for the supercritical case.

For the subcritical case, i.e. $0 < m < 1$ and $0 < h'\left(1\right) = \mu < \infty$, it is known that $Z\left(t\right)$ has a proper limit distribution that is

$$\lim_{t \to \infty} P\left(Z\left(t\right) = k\right) = \rho_k, \quad k = 0, 1, 2, \ldots \tag{7}$$

exist, where $\left\{\rho_k, k \geq 0\right\}$ is a probability distribution and the generating function of this stationary distribution is

$$g\left(u\right) = h\left(u\right) \prod_{t=1}^{\infty} h\left(f_t\left(u\right)\right)$$

and $f_t\left(u\right)$ is the generating function of X_{ti} at time t.

Theorem 3: If $0 < m < 1$, then

$$\lim_{t \to \infty} P\left(S_{Z(t)}^{W} = j\right) = r_j$$

where r_j is a probability distribution with generating function

$$\sum_{j=0}^{\infty} r_j \mu^j = E\left[g\left(1 - \Delta\left(x\right) + \Delta\left(x\right) u\right)\right]$$

Proof. We obtain the generating function of $S_{Z(t)}^{W}$,

$$E\left(\mu^{S_{Z(t)}^{W}}\right) = E\left\{\sum_{k=0}^{\infty} \sum_{j=0}^{k} u^j \binom{k}{j} \left[\Delta\left(x\right)\right]^j \left[1 - \Delta\left(x\right)\right]^{k-j} P(Z\left(t\right) = k \mid Z\left(t\right) > 0\right\}$$

$$= E\left\{\sum_{k=0}^{\infty}\left[1-\Delta(x)+\Delta(x)\mu\right]^k P(Z(t)=k \mid Z(t)>0\right\}$$

$$\to E\left\{\left[1-\Delta(x)+\Delta(x)\mu\right]^z\right\},$$

taking into account of (7).

4. ASYMPTOTICS OF MOMENTS

We conclude the discussion with the remarks on the asymptotic behavior of the same moment of the process $S_{Z(t)}^W$. Using (5), we obtain the expected value of $S_{Z(t)}^W$:

$$E\left[S_{Z(t)}^W\right] = EE\left\{\sum_{j=0}^{Z(t)} j\binom{Z(t)}{j}\left[\Delta(x)\right]^j\left[1-\Delta(x)\right]^{Z(t)-j} \mid Z(t),W\right\}$$

$$= EE\left\{Z(t)\Delta(x)\middle|Z(t),W\right\}$$

$$= E\left\{Z(t)\Delta(x)\right\}$$

Assuming that $Z(t)$ and are independent, differentiating (1), we obtain

$$E\left(Z(t)\right) = \mu t + 1$$

Thus,

$$E\left[S_{Z(t)}^W\right] = E\left[Z(t)\right]E\left[\Delta(x)\right] = (\mu t + 1)E\Delta(x)$$

It can be shown that for $m = 1$

$$Var\left[Z(t)\right] = Var\left[Z(t-1)\right] + \mu\sigma^2(t-1) + \sigma^2 + \tau^2$$

$$= t\left(\sigma^2 + \tau^2\right) + \frac{t(t-1)}{2}\mu\sigma^2.$$

The variance of the process then,

$$Var\left[S_{Z(t)}^W\right] = EZ(t)\left(Z(t)-1\right)[\Delta(x)]^2 + E[Z(t)]\Delta(x) - (ES_{Z(t)}^W)^2$$

$$= Var\left\{Z(t)\Delta(x)\right\} + E\left\{Z(t)\Delta(x)\left[1-\Delta(x)\right]\right\}$$

which simplify to

$$(\mu t + 1)E\Delta(x)\left[1-(\mu t+1)E\Delta(x)\right]$$

$$+\left[\left(t\left(\sigma^2+\tau^2\right)+\frac{t(t-1)}{2}\mu\sigma^2\right)+(\mu t+1)(\mu t)\right]E\left(\Delta(x)\right)^2.$$

Assuming $m \neq 1,\ Z(t)$ and W are independent, it can be shown that

$$EZ(t) = \mu\frac{1-m^t}{1-m} + m^t$$

and $VarZ(t) = m^{2(t-1)}\left(\sigma^2+\tau^2\right)+\left(\tau^2-\mu+\mu^2\right)\frac{1-m^{2(t-1)}}{1-m^2} + \sum_{i=0}^{t-2} m^{2i} S_{t-i}$

where

$$S_t = EZ(t-1)\left(m^2 EZ(t-1)+\sigma^2-\mu+2\mu m\right)+\left(1-EZ(t)\right)EZ(t)$$

Thus,

$$E\left(S^W_{Z(t)}\right) = EZ(t)\,E\Delta(x) \text{ and}$$

$$VarS^W_{Z(t)} = \left(\mu\frac{1-m^t}{1-m}+m^t\right)E\Delta(x)\left[1-\left(\mu\frac{1-m^t}{1-m}+m^t\right)E\Delta(x)\right]$$

$$+\left\{VarZ(t)+\left(\mu\frac{1-m^t}{1-m}+m^t\right)\left(\mu\frac{1-m^t}{1-m}+m^t-1\right)\right\}E[\left(\Delta(x)\right)^2].$$

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