UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MSS 302 – Real Analysis
[Analisis Nyata]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

...2/-

1. Give the definitions of

   (a) Outer measure
   (b) Lebesgue measurable set
   (c) Measurable function
   (d) Lebesgue integral for a simple function
   (e) Lebesgue integral for a non-negative function
   (f) Lebesgue integral for an arbitrary function

[6 marks]

2. Prove that all irrational numbers in $[0,1]$ is measurable, and find its measure.

[7 marks]

3. Answer the following question, and give reasons:

   (a) Is the set $\cup_{n=1}^{\infty}\left\{x\in\mathbb{R}:0<x<\frac{1}{3^n}\right\}$ measurable?

   (b) Is the set $\cap_{n=1}^{\infty}\left\{x\in\mathbb{R}:0<x<\frac{1}{3^n}\right\}$ measurable?

   If so, then find their measures.

[10 marks]

4. Let $f:[0,1]\rightarrow\mathbb{R}$ be a function defined by $f(x)=\begin{cases}\cos x, & \text{jika } x\in\mathbb{Q}\\ \sin x, & \text{jika } x\notin\mathbb{Q}\end{cases}$.

   Is f Lebesgue integrable? Justify your answer.

[10 marks]

5. Given a sequence of functions (f_n) where $f_n(x)=(x+\frac{1}{n})^{-\frac{1}{4}}$ on the interval $[0,1]$.

   (a) Find the pointwise limit f of the sequence (f_n) on $[0,1]$.

   (b) Show that f is measurable on $[0,1]$.

   (c) Then find the integral $\int_{[0,1]} f$.

[12 marks]

6. (a) Provide an example of a sequence (f_n) of measurable functions on $[0,1]$ such that $f_n\rightarrow f$ almost everywhere, and $f_n\geq 0$, but

   $$\liminf_{n\rightarrow\infty}\int_{[0,1]} f_n \neq \int f.$$

   (b) Examine which one of the three theorems: Monotone Convergence Theorem, Fatou's Lemma, Dominated convergence Theorem, is relevant to your example in part (a).

[12 marks]

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1. *Beri takrif bagi*

   (a) *Sukatan terkeluar*
   (b) *Set tersukatkan Lebesgue*
   (c) *Fungsi tersukatkan*
   (d) *Kamiran Lebesgue untuk fungsi ringkas*
   (e) *Kamiran Lebesgue untuk fungsi bukan negative*
   (f) *Kamiran Lebesgue untuk fungsi sebarangan*

[6 markah]

2. *Buktikan bahawa semua nombor tak nisbah dalam* $[0,1]$ *adalah tersukatkan, dan cari sukatannya.*

[7 markah]

3. *Jawab soalan berikut, dan beri penjelasan:*

   (a) *Adakah set* $\cup_{n=1}^{\infty}\left\{x \in \mathbb{R} : 0 < x < \frac{1}{3^n}\right\}$ *tersukatkan?*

   (b) *Adakah set* $\cap_{n=1}^{\infty}\left\{x \in \mathbb{R} : 0 < x < \frac{1}{3^n}\right\}$ *tersukatkan?*
   Jika ya, maka cari sukatannya.

[10 markah]

4. *Biarkan* $f:[0,1] \to \mathbb{R}$ *suatu fungsi ditakrif sebagai* $f(x) = \begin{cases} \cos x, & \text{jika } x \in \mathbb{Q} \\ \sin x, & \text{jika } x \notin \mathbb{Q} \end{cases}$.
Adakah f *terkamirkan cara Lebesgue? Terangkan jawapan anda.*

[10 markah]

5. *Diberi suatu jujukan fungsi* (f_n) *dengan* $f_n(x) = (x + \frac{1}{n})^{-\frac{1}{4}}$ *pada selang* $[0,1]$.

   (a) *Cari had titik demi titik* f *untuk jujukan* (f_n) *pada* $[0,1]$.
   (b) *Tunjukkan bahawa* f *tersukatkan pada* $[0,1]$.
   (c) *Kemudian cari kamiran* $\int_{[0,1]} f$.

[12 markah]

6. (a) *Beri satu contoh jujukan* (f_n) *fungsi tersukatkan pada* $[0,1]$ *sedemikian* $f_n \to f$ *hampir di mana-mana, dan* $f_n \geq 0$, *tetapi*

$$\liminf_{n\to\infty} \int_{[0,1]} f_n \neq \int f.$$

   (b) *Selidik mana di antara tiga teorem berikut: Teorem Pertumpuan Berekanada, Lema Fatou, Teorem Pertumpuan Terdominasi, relevan dengan contoh anda di bahagian (a).*

[12 markah]

...4/-

7. Suppose $f:\mathbb{R}\to[0,\infty)$ is a measurable function, and $k\neq 0$ is a real number. Let $g(x)=f(kx)$ untuk $x\in\mathbb{R}$. Show that g is a measurable function, and then prove that $\int g=\frac{1}{|k|}\int f$.

[12 marks]

8. State precisely what it means by the statement: "elements of $L^p(X)$ are not functions but equivalence classes of functions".

[5 marks]

9. Prove/disprove: $L^p(X)\subseteq L^q(X)$ whenever $1<q<p<\infty$ and $m(X)<\infty$.

[12 marks]

10. Give, with proof, an example $f\in L^1[0,1]$, but is not in $L^2[0,1]$.

[14 marks]

7. *Andaikan* $f:\mathbb{R}\to[0,\infty)$ *fungsi tersukatkan, dan* $k\neq 0$ *suatu nombor nyata. Biarkan* $g(x)=f(kx)$ *untuk* $x\in\mathbb{R}$. *Tunjukkan bahawa g fungsi tersukatkan, dan tunjukkan bahawa* $\int g=\frac{1}{|k|}\int f$.

[12 markah]

8. *Nyatakan dengan tepat makna untuk pernyataan berikut: "Unsur untuk* $L^p(X)$ *adalah bukan fungsi tetapi kelas kesetaraan untuk fungsi".*

[5 markah]

9. *Bukti/Sangkal:* $L^p(X)\subseteq L^q(X)$ *untuk* $1<q<p<\infty$ *dan* $m(X)<\infty$.

[12 markah]

10. *Beri dengan bukti, suatu contoh sedemikian* $f\in L^1[0,1]$, *tetapi bukan dalam* $L^2[0,1]$.

[14 markah]

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