UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

...2/-

1. (a) Let $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, be the probability density function of X. Find the cumulative distribution function of $Y = X^2$. Hence, find the probability density function for Y.

[30 marks]

(b) Let X have a Poisson distribution with parameter λ. Let the conditional probability mass function of Y given $X = n$ be $P(Y = y \mid X = n) = \binom{n}{y} p^y (1-p)^{n-y}$. Find $E(Y)$.

(Hint: Find $E(Y \mid X = n)$ first)

[20 marks]

(c) Let X have the probability density function $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere. Find the cumulative distribution function and the probability density function of $Y = -2\ln X^4$.

[20 marks]

(d) The joint discrete random variables (X, Y) have probability values as shown below:

$P(X=0, Y=0) = \frac{4}{25}$ $P(X=0, Y=1) = \frac{6}{25}$ $P(X=1, Y=0) = \frac{6}{25}$

$P(X=1, Y=1) = \frac{9}{25}$

Find $E(X|Y=0)$ and $E(X|Y=1)$

[30 marks]

2. (a) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size 4 from the distribution having probability density function

$$f(x) = e^{-x}, 0 < x < \infty, \text{ zero elsewhere. Find } P(3 \le Y_4).$$

[30 marks]

(b) Differentiate the moment-generating function for the geometric random variable and verify the expressions given for $E(X) = (1-p)/p$ and $Var(X) = (1-p)/p^2$.

[20 marks]

(c) (i) Let U have an F distribution with parameters r and s. Using the related theorem, prove that $\frac{1}{U}$ have an F distribution with parameters s and r.

(ii) Let $T = \frac{W}{\sqrt{V/r}}$, where the independent variables W and V follow the standard normal distribution and the chi-square distribution with r degrees of freedom, respectively. Using the related theorem, show that T^2 has an F distribution with parameters 1 and r.

[30 marks]

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1. (a) *Andaikan* $f(x) = \frac{1}{3}$, $-1 < x < 2$, *sifar di tempat lain, adalah fungsi ketumpatan kebarangkalian bagi X. Cari fungsi taburan longgokan bagi* $Y = X^2$. *Seterusnya, cari fungsi ketumpatan kebarangkalian Y.*

[30 markah]

(b) *Andaikan X mempunyai taburan Poisson dengan parameter* λ. *Biarkan fungsi jisim kebarangkalian bersyarat bagi Y diberi X = n sebagai* $\mathrm{P}(Y = y \mid X = n) = \binom{n}{y} p^y (1-p)^{n-y}$. *Cari E(Y).*

(Petunjuk: Cari $E(Y \mid X = n)$ *dahulu)*

[20 markah]

(c) *Andaikan X mempunyai fungsi ketumpatan kebarangkalian* $f(x) = 4x^3$, $0 < x < 1$, *sifar di tempat lain. Cari fungsi taburan longgokan dan fungsi ketumpatan kebarangkalian bagi* $Y = -2\ln X^4$.

[20 markah]

(d) *Pembolehubah rawak diskret tercantum (X, Y) mempunyai nilai kebarangkalian seperti yang ditunjukkan di bawah:*

$P(X=0, Y=0) = \frac{4}{25}$ $P(X=0, Y=1) = \frac{6}{25}$ $P(X=1, Y=0) = \frac{6}{25}$

$P(X=1, Y=1) = \frac{9}{25}$

Cari $E(X|Y=0)$ *dan* $E(X|Y=1)$

[30 markah]

2. (a) *Andaikan* $Y_1 < Y_2 < Y_3 < Y_4$ *adalah statistik tertib bagi suatu sampel rawak saiz 4 daripada suatu taburan dengan fungsi ketumpatan kebarangkalian* $f(x) = e^{-x}, 0 < x < \infty$, *sifar di tempat lain. Cari* $P(3 \le Y_4)$.

[30 markah]

(b) *Bezakan fungsi penjadi momen bagi pembolehubah rawak geometri dan tunjukkan ungkapan yang diberi bagi* $E(X) = (1-p)/p$ *dan* $Var(X) = (1-p)/p^2$.

[20 markah]

(c) (i) *Biarkan U mempunyai taburan F dengan parameter r dan s. Dengan menggunakan teorem yang berkaitan, buktikan bahawa* $\frac{1}{U}$ *mempunyai taburan F dengan parameter s dan r.*

(ii) *Biarkan* $T = \frac{W}{\sqrt{V/r}}$, *yang mana pembolehubah tak bersandar W dan V, masing-masing mengikuti taburan normal piawai dan taburan khi kuasa dua dengan r darjah kebebasan. Dengan menggunakan teorem yang berkaitan, tunjukkan bahawa* T^2 *mempunyai taburan F dengan parameter 1 dan r.*

[30 markah]

...4/-

(d) Let the probability mass function of Y_n be $P(Y_n = n) = 1$. Show that Y_n does not have a limiting distribution.

[20 marks]

3. (a) Suppose a random sample, $Y_1, Y_2, ..., Y_n$, is taken from a $N(\mu, \sigma^2)$ distribution, where σ^2 is known.

(i) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of μ.

(ii) Is $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$ an efficient estimator for μ?

[30 marks]

(b) Given that $Y_1 = 2.3$, $Y_2 = 1.9$ and $Y_3 = 4.6$ is a random sample from a distribution with probability density function (pdf)

$$f_Y(y;\theta) = \frac{y^3 e^{-y/\theta}}{6\theta^4}, \ y \geq 0.$$

Calculate the maximum likelihood estimate for θ.

[20 marks]

(c) Let $\bar{X}$ and $\bar{Y}$ be the means of two independent samples, each sample of size n and having the $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ distributions, respectively, where the common variance, σ^2 is known. Find n so that

$$P\left(\bar{X} - \bar{Y} - \frac{\sigma}{5} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \frac{\sigma}{5}\right) = 0.90.$$

[20 marks]

(d) (i) Define the pivotal quantity.

(ii) Assume that $X_1, X_2, ..., X_n$ is a random sample having a $N(5, \sigma^2)$ distribution, where σ^2 is unknown. Is the random variable $\frac{\bar{X}}{\sqrt{\sigma}}$ a pivotal quantity? Explain.

[30 marks]

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(d) Biarkan fungsi jisim kebarangkalian bagi Y_n sebagai $P(Y_n = n) = 1$. Tunjukkan bahawa Y_n tidak mempunyai taburan penghad.

[20 markah]

3. *(a) Andaikan suatu sampel rawak, $Y_1, Y_2, \ldots, Y_n$, diambil daripada taburan $N(\mu, \sigma^2)$, yang mana σ^2 diketahui.*

   (i) Cari batas bawah Cramer-Rao untuk varians penganggar saksama μ.

   (ii) Adakah $\bar{Y} = \dfrac{\sum_{i=1}^{n} Y_i}{n}$ suatu penganggar cekap bagi μ?

[30 markah]

(b) Diberi $Y_1 = 2.3$, $Y_2 = 1.9$ dan $Y_3 = 4.6$ suatu sampel rawak daripada taburan dengan fungsi ketumpatan kebarangkalian (fkk)

$$f_Y(y;\theta) = \frac{y^3 e^{-y/\theta}}{6\theta^4}, \; y \geq 0.$$

Kirakan anggaran kebolehjadian maksimum bagi θ.

[20 markah]

(c) Biarkan $\bar{X}$ dan $\bar{Y}$ sebagai min bagi dua sampel tak bersandar, setiap sampel dengan saiz n dan masing-masing mempunyai taburan $N(\mu_1, \sigma^2)$ dan $N(\mu_2, \sigma^2)$, yang mana varians sepunya, σ^2 diketahui. Cari n supaya

$$P\left(\bar{X} - \bar{Y} - \frac{\sigma}{5} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \frac{\sigma}{5}\right) = 0.90.$$

[20 markah]

(d) (i) Takrifkan kuantiti pangsian.

   (ii) Andaikan $X_1, X_2, \ldots, X_n$ suatu sampel rawak yang mempunyai taburan $N(5, \sigma^2)$, yang mana σ^2 tidak diketahui. Adakah pembolehubah rawak $\dfrac{\bar{X}}{\sqrt{\sigma}}$ suatu kuantiti pangsian? Jelaskan.

[30 markah]

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4. (a) Let Y have a binomial distribution with parameters n and p. We reject $H_0 : p = \frac{1}{2}$ and accept $H_1 : p > \frac{1}{2}$ if $Y \geq c$. Find n and c that gives a power function $\pi(p)$, such that $\pi\left(\frac{1}{2}\right) = 0.10$ and $\pi\left(\frac{2}{3}\right) = 0.95$, approximately.

[20 marks]

(b) If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution having pdf $f(x;\theta) = \theta x^{\theta-1}$, $0<x<1$, zero elsewhere, show using the Neyman-Pearson Theorem that the best critical region for testing $H_0 : \theta = 1$ vs. $H_1 : \theta = 2$ is $C = \left\{(x_1, x_2, \ldots, x_n) : c \leq \prod_{i=1}^{n} x_i\right\}$.

[20 marks]

(c) Based on a random sample of size 10 from a $N(0, \sigma^2)$ distribution, find a uniformly most powerful (UMP) test of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 > 1$.

[30 marks]

(d) Assume that $X_1, X_2, \ldots, X_n$ is a random sample from a $N(0, \theta)$ distribution, where $\theta > 0$. Find the likelihood ratio (LR) test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.

[30 marks]

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4. (a) *Biarkan Y mempunyai taburan binomial dengan parameter n dan p. Kita menolak* $H_0 : p = \frac{1}{2}$ *dan menerima* $H_1 : p > \frac{1}{2}$ *jika* $Y \geq c$. *Cari n dan c yang memberikan fungsi kuasa* $\pi(p)$, *sedemikian sehingga* $\pi\left(\frac{1}{2}\right) = 0.10$ *dan* $\pi\left(\frac{2}{3}\right) = 0.95$, *secara hampiran.*

[20 markah]

(b) *Jika* $X_1, X_2, ..., X_n$ *ialah sampel rawak daripada suatu taburan yang mempunyai fkk* $f(x;\theta) = \theta x^{\theta-1}$, $0<x<1$, *sifar di tempat lain, tunjukkan dengan menggunakan Teorem Neyman-Pearson bahawa rantau genting terbaik untuk menguji* $H_0 : \theta = 1$ *lawan* $H_1 : \theta = 2$ *ialah* $C = \left\{(x_1, x_2, ..., x_n) : c \leq \prod_{i=1}^{n} x_i \right\}$.

[20 markah]

(c) *Berdasarkan sampel rawak saiz 10 daripada taburan* $N(0, \sigma^2)$, *cari ujian paling berkuasa secara seragam (UPBS) saiz* $\alpha = 0.05$ *untuk menguji* $H_0 : \sigma^2 = 1$ lawan $H_1 : \sigma^2 > 1$.

[30 markah]

(d) *Andaikan* $X_1, X_2, ..., X_n$ *ialah sampel rawak daripada taburan* $N(0, \theta)$, *yang mana* $\theta > 0$. *Cari ujian nisbah kebolehjadian (NK) bagi menguji* $H_0 : \theta = \theta_0$ lawan $H_1 : \theta > \theta_0$.

[30 markah]

...8/-

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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