UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 514 – Mathematical Modeling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

[Arahan: *Jawab **semua lima** [5] soalan.]*

...2/-

1. (a) By making all necessary assumptions, derive the logistic population model

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

with growth-rate k and carrying capacity N.

(b) Consider the population model

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{230}\right),$$

where $P(t)$ is the population at time t.

(i) For what values of P is the population in equilibrium?
(ii) For what values of P is the population increasing?
(iii) For what values of P is the population decreasing?

(c) A lake has a carrying capacity of 300 fishes. On July 1, 2000, the lake is stocked with 50 fishes. Twelve months later there are 100 fishes in the lake. Assuming a logistic fish population, in what month of what year will there be 200 fishes in the lake?

[15 marks]

2. (a) Derive an expression for the damped amplitude, the damped natural frequency and the damped period of the damped vibrations modeled by equation

$$mx'' + ax' + kx = 0.$$

(b) Determine whether the motion described by the given differential equation is underdamped, critically damped, or over damped.

(i) $x'' + 4x = 0$
(ii) $x'' - 2x' + x = 0$
(iii) $x'' + 4x' + 4x = 0$

(c) An 8-lb weight is attached to a spring, which in turn is suspended from the ceiling. The weight comes to rest in equilibrium position, stretching the spring 6 inches in the process. The weight is then pulled down an addition 1 ft and released with an upward velocity of 8 ft/sec. Assume that the resistance of the medium is negligible.

(i) Formulate the mathematical model that describes the motion of the weight.
(ii) Find the motion of the weight.
(iii) Find the amplitude, phase angle, frequency and period of the weight.

[15 marks]

...3/-

1. (a) *Dengan membuat semua andaian yang perlu,, terbitkan model populasi logistik*

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

dengan kadar pertumbuhan k dan kapasiti pembawaan N.

(b) *Pertimbangkan model populasi*

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{230}\right),$$

yang mana P(t) adalah populasi pada masa t.

(i) *Apakah nilai P dengan populasi dalam keadaan seimbang?*
(ii) *Apakah nilai P dengan populasi bertambah?*
(iii) *Apakah nilai P apabila populasi menurun?*

(c) *Sebuah tasik mempunyai kapasiti pembawaan 300 ekor ikan. Pada 1 Julai 2000, tasik tersebut diisi dengan 50 ekor ikan. Dua belas bulan kemudian, terdapat sebanyak 100 ekor ikan di dalam tasik tersebut. Dengan anggapan model populasi ikan adalah logistik, dalam bulan dan tahun apakah akan terdapat 200 ekor ikan di dalam tasik?*

[15 markah]

2. (a) *Terbitkan ungkapan untuk amplitud lembapan, kekerapan asli lembapan dan kalaan lembapan bagi getaran lembapan yang dimodel oleh persamaan.*

$$m x'' + a x' + kx = 0.$$

(b) *Tentukan samada pergerakan yang diperihalkan oleh persamaan pembezaan di bawah adalah lembapan kurangan, lembapan genting atau terlampau lembapan.*

(i) $x'' + 4x = 0$
(ii) $x'' - 2x' + x = 0$
(iii) $x'' + 4x' + 4x = 0$

(c) *Sebuah pemberat 8 paun dilekatkan pada suatu spring yang kemudiannya digantung daripada siling. Pemberat tersebut berehat pada kedudukan seimbang, sambil meregangkan spring sebanyak 6 inci dalam proses tersebut. Pemberat tersebut kemudian ditarik ke bawah dengan tambahan 1 kaki dan dilepaskan naik dengan kelajuan 8 kaki/saat. Andaikan rintangan medium boleh diabaikan.*

(i) *Rumuskan model matematik yang memperihalkan pergerakan pemberat.*
(ii) *Cari pergerakan pemberat.*
(iii) *Cari amplitud, sudut fasa, kekerapan dan kalaan pemberat.*

[15 markah]

...4/-

3. (a) The temperature during a typical late July day in Malaysia varies periodically between a minimum of 70° F at 4 AM and a maximum of 90 ° F at 4 PM.

(i) Find a trigonometric model giving the temperature as a function of time t, the number of hours past midnight.
(ii) Use this model to predict the temperatures at midnight and at noon during the day.
(iii) Find when during the afternoon the temperature is exactly 87.5 ° F.

(b) A company manufacturing computers plans to introduce two new products. Both computers contain the same microprocessing chip, but one system is equipped with a 27-in. monitor and the other system with a 31-in. monitor. In addition to RM 400,000 in fixed cost, it costs the company RM 1950 to produce a 27-in. system and RM 2250 to produce a 31-in system. The manufacturer's suggested retail price is RM 3390 for the 27-in. system and RM 3990 for the 31-in. system. In the competitive market in which the systems will be sold, the marketing staff estimates that for each additional system for a particular type sold, the selling price will fall by RM 0.10. Furthermore, sales of each type system affect the sales of the other: It is estimated that the selling price for each 27-in. system is reduced by RM 0.03 for each 31-in. computer sold, and the selling price for each 31-in. system is reduced by RM 0.04 for each 27-in. computer sold. Assuming they can sell all the computers they make:

Formulate a mathematical model and solve for how many systems of each type the company should manufacture to maximize its profit.

[20 marks]

4. (a) What are viscous and non-viscous fluids? Derive Euler's equations of motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

of a non-viscous fluid.

(b) Explain the concept of shallow water theory. State clearly all basic assumptions of shallow water theory. Establish the depth averaged shallow water equations.

[25 marks]

3. (a) *Suhu suatu hari tipikal pada lewat Julai di Malaysia berubah secara berkala di antara minimum 70° F pada jam 4:00 pagi dan maksimum 90 ° F pada jam 4:00 petang.*

(i) Cari model trigonometri yang menyatakan/ mengungkapkan suhu sebagai suatu fungsi masa t, iaitu bilangan jam selepas tengah malam.

(ii) Guna model ini untuk meramalkan suhu pada tengah malam dan tengah hari di waktu siang.

(iii) Cari masa pada waktu petang ketika suhu adalah dengan tepat 87.5 ° F.

(b) *Sebuah syarikat pengeluaran komputer bercadang memperkenalkan dua produk baru. Kedua-dua komputer mempunyai cip mikroprosesor yang sama tetapi satu sistem dilengkapi dengan monitor berukuran 27 inci dan (yang satu lagi) dengan monitor 31 inci. Di samping kos tetap sebanyak RM 400,000, RM 1950 juga diperlukan untuk menghasilkan sistem 27 inci dan RM 2250 diperlukan untuk menghasilkan sistem 31 inci. Pengeluar mencadangkan harga pasaran RM 3390 untuk sistem 27 inci dan RM 3990 untuk sistem 31 inci. Dalam pasaran yang bersaing dimana sistem akan dijual, pegawai pemasaran menganggarkan bahawa untuk setiap sistem tambahan bagi setiap jenis yang dijual, harga pasaran akan jatuh sebanyak RM0.10. Tambahan pula, setiap jualan sistem akan memberi kesan ke atas jualan sistem yang satu lagi. Dianggarkan harga jualan untuk setiap sistem 27 inci akan dikurangkan sebanyak RM0.03 untuk setiap komputer 31 inci yang dijual, dan harga jualan untuk setiap sistem 31 inci akan berkurang sebanyak RM0.04 untuk setiap komputer 27 inci yang dijual. Andaikan semua komputer yang dihasilkan syarikat dapat dijual:*

Rumuskan suatu model matematik dan selesaikan untuk mendapatkan bilangan sistem bagi setiap jenis yang patut keluarkan oleh syarikat untuk memaksimumkan keuntungan.

[20 markah]

4. (a) *Apakah bendalir likat dan tak likat? Terbitkan persamaan gerakan Euler*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

bagi bendalir tak likat.

(b) *Huraikan konsep teori air cetek. Nyatakan dengan jelas semua andaian asas teori air cetek. Dapatkan persamaan purata kedalaman air cetek.*

[25 markah]

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5. (a) Explaining all the basic assumptions derive the following fundamental one-dimensional mass transport equation

$$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial x} = r_g - r_d$$

for a substance well mixed with a fluid. Hence write down the three-dimensional form of the equation.

(b) A tank with a fixed volume of 100 L has water flowing through an inlet and an outlet at a constant rate of 5 L/min. The water in the tank is continuously mixed by a mechanical device. The concentration of dissolved oxygen (DO) in the tank is initially zero, and the inlet flow is at the saturation concentration C_s of DO.

(i) Formulate the mathematical models for the following cases:
   (a) The tank is open to the atmosphere so that oxygen can transfer across the air-water interface.
   (b) The tank is completely sealed with oxygen entering the system only through the inlet.

(ii) Solve the models.

(iii) Determine the concentration of DO in the tank 10 min after initiation of the flow under two conditions.
Assume that the oxygen flux occurs at the air-water interface according to the relation $r = k(C_s - C)$. Given data are $C_s = 9$ mg/L and $k = 0.004\ \text{s}^{-1}$.

[25 marks]

5. (a) *Dengan menerangkan semua andaian asas, terbitkan persamaan pengangkutan jisim satu dimensi asasi yang berikut*

$$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial x} = r_g - r_d$$

untuk suatu bahan yang bercampur rapi dengan bendalir. Seterusnya tulis persamaan tersebut dalam bentuk tiga-dimensi.

(b) *Suatu tangki dengan isipadu tetap 100 L mempunyai air yang mengalir melalui suatu saluran masuk dan suatu saluran keluar pada kadar malar 5 L/min. Air di dalam tangki dicampur secara berterusan oleh suatu alat mekanikal. Kepekatan awal oksigen terlarut (DO) dalam tangki adalah sifar, dan aliran masuk mempunyai kepekatan tepu terlarut C_s untuk DO.*

(i) *Rumuskan model matematik untuk kes-kes berikut :*
   (a) *Tangki dibuka kepada atmosfera supaya oksigen boleh berpindah melalui permukaan udara-air.*
   (b) *Tangki ditutup sepenuhnya dengan oksigen memasuki sistem hanya melalui saluran masuk.*

(ii) *Selesaikan kedua-dua model.*

(iii) *Tentukan kepekatan DO di dalam tangki 10 minit selepas permulaan pengaliran di bawah dua keadaan.*
Andaikan fluks oksigen mengikut kedua-dua keadaan permukaan udara-air mengikut hubungan $r = k(C_s - C)$. Data yang diberikan ialah C_s = 9 mg/L dan $k = 0.004\,s^{-1}$.

[25 markah]

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