UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. Consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u$$

(a) Write the Forward Time, Centred Space scheme for this equation

(b) Analyze the consistency of this scheme for the above equation.

[100 marks]

2. Consider the one-dimensional inviscid Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \text{ where } u>0.$$

(a) Write the Forward Time, Backward Space scheme for this equation

(b) By treating u in $u\dfrac{\partial u}{\partial x}$ as a constant, analyze the stability of this scheme.

[100 marks]

3. (a) Consider the boundary value problem

$$u_{xx}+u_{yy}=0, \qquad 0\le x\le \pi, \quad 0\le y\le \pi$$

with the values of u specified at the boundaries. Assume that the boundary value problem above is discretised using the five-point formula with mesh size n = 45 and $\Delta x=\Delta y=h$.

(i) Find the spectral radius $\rho(B)$ of the Jacobi point iterative method.

(ii) What is the spectral radius of the Gauss-Seidel point iterative method?

(iii) What is the theoretical optimal relaxation parameter ω_b for the point S.O.R.?

(iv) What is the spectral radius $\rho(L_{\omega_b})$, where L_{ω_b} is the S.O.R. iteration matrix?

(v) Based on the five-point formula, estimate the number of iterations you would expect to get if the Jacobi, Gauss-Seidel and S.O.R. methods are used for this mesh size and tolerance $\varepsilon=10^{-7}$.

...3/-

1. *Pertimbangkan persamaan*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u$$

(a) *Tuliskan skema beza ke depan dalam masa dan beza ke pusat dalam ruang untuk persamaan ini.*

(b) *Analisiskan kekonistenaan skema ini untuk persamaan di atas.*

[100 markah]

2. *Pertimbangkan Persamaan Burger satu dimensi tak likat*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ dengan } u > 0.$$

(a) *Tuliskan skema beza ke depan dalam masa dan beza ke belakang dalam ruang untuk persamaan ini.*

(b) *Dengan menganggap* ***u*** *dalam* $\boldsymbol{u}\frac{\partial u}{\partial x}$ *sebagai suatu pemalar analisiskan kestabilan skema ini.*

[100 markah]

3. (a) *Pertimbangkan masalah nilai sempadan*

$$u_{xx} + u_{yy} = 0, \qquad 0 \le x \le \pi, \quad 0 \le y \le \pi$$

dengan nilai-nilai u dispesifikasikan pada sempadan-sempadannya. Anggapkan masalah nilai sempadan di atas didiskretkan menggunakan rumus lima-titik dengan saiz mesh n = 45 dan $\Delta x = \Delta y = h$.

(i) *Cari jejari spektrum* $\rho(B)$ *bagi kaedah lelaran titik Jacobi.*
(ii) *Apakah jejari spektrum bagi kaedah lelaran titik Gauss-Seidel?*
(iii) *Apakah parameter pengenduran optimum teoretikal* ω_b *bagi S.O.R. titik?*
(iv) *Apakah jejari spektrum* $\rho(L_{\omega_b})$, *di mana* L_{ω_b} *ialah matriks lelaran S.O.R.?*
(v) *Berdasarkan kepada rumus lima-titik, anggarkan bilangan lelaran yang anda jangka akan diperoleh jika kaedah-kaedah Jacobi, Gauss-Seidel dan S.O.R. digunakan untuk saiz mesh ini dan tolerans* $\varepsilon = 10^{-7}$.

...4/-

(b) Consider the system $A\underline{u} = \underline{b}$ where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using a preconditioned system

$$M^{-1}A\underline{u} = M^{-1}\underline{b},$$

generate 2 iteration values, $\underline{u}^{(1)}$, $\underline{u}^{(2)}$, of the preconditioned Conjugate Gradient method with preconditioner matrix $M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

[100 marks]

4. (a) Consider the Laplace equation

$$\nabla^2 u = 0$$

on the region $0 \le x \le 1,\ 0 \le y \le 1$ with boundary conditions

$$u_x(0,y) = 0, \qquad u(x,0) = 0$$
$$u_x(1,y) = 1, \qquad u(x,1) = 5$$

(i) Discretise the above pde using the five-point centred difference approximations with h = 1/3 in row-wise natural ordering and derive the resulting system $A\underline{u} = \underline{b}$ in 8 equations and 8 unknowns.

(ii) Write the Successive OverRelaxation (S.O.R.) iterative formula in solving this system.

(b) In solving the system $A\underline{u} = \underline{b}$ using the second order Richardson's method, we use the formula

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha(\underline{b} - A\underline{u}^{(k)}) + \beta(\underline{u}^{(k)} - \underline{u}^{(k-1)})$$

where α and β are constants chosen to provide maximum convergence to the solution. If λ_i are the eigenvalues of A and γ_i are the eigenvalues of the matrix associated with the iterative process, show that λ_i and γ_i are related by the following equation:

$$\gamma_i = \frac{(1+\beta-\alpha\lambda_i) \pm \sqrt{(1+\beta-\alpha\lambda_i)^2 - 4\beta}}{2}$$

[100 marks]

...5/-

(b) *Pertimbangkan sistem* $A\underline{u} = \underline{b}$ *di mana*

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Dengan menggunakan sistem berprasyarat

$$M^{-1}A\underline{u} = M^{-1}\underline{b},$$

janakan 2 nilai lelaran, $\underline{u}^{(1)}$, $\underline{u}^{(2)}$, *bagi kaedah Kecerunan Konjugat berprasyarat dengan matriks prasyarat* $M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

[100 markah]

4. (a) *Pertimbangkan persamaan Laplace*

$$\nabla^2 u = 0$$

pada rantau $0 \le x \le 1,\ 0 \le y \le 1$ *dengan syarat sempadan*

$$u_x(0,y) = 0, \qquad u(x,0) = 0$$
$$u_x(1,y) = 1, \qquad u(x,1) = 5$$

(i) *Diskretkan persamaan pembezaan di atas menggunakan anggaran beza ketengah lima-titik dengan* $h = 1/3$ *dalam tertib baris semulajadi dan terbitkan sistem yang terhasil* $A\underline{u} = \underline{b}$ *dalam 8 persamaan dan 8 pembolehubah.*

(ii) *Tuliskan rumus lelaran Pengenduran Berlebihan Berturut-turut (S.O.R.) dalam menyelesaikan sistem ini.*

(b) *Dalam menyelesaikan sistem* $A\underline{u} = \underline{b}$ *menggunakan kaedah Richardson peringkat dua, kita gunakan rumus*

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} + \alpha(\underline{b} - A\underline{u}^{(k)}) + \beta(\underline{u}^{(k)} - \underline{u}^{(k-1)})$$

di mana α *dan* β *adalah pemalar-pemalar terpilih untuk memperoleh penumpuan maksimum kepada penyelesaiannya. Jika* λ_i *adalah nilai-nilai eigen bagi A dan* γ_i *adalah nilai-nilai eigen bagi matriks yang bersepadan dengan proses lelaran, tunjukkan bahawa* λ_i *dan* γ_i *adalah berkait melalui persamaan berikut:*

$$\gamma_i = \frac{(1+\beta-\alpha\lambda_i) \pm \sqrt{(1+\beta-\alpha\lambda_i)^2 - 4\beta}}{2}$$

[100 markah]

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