UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MSG 388 – Mathematical Algorithms for Computer Graphics
[Algoritma Matematik untuk Grafik Komputer]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

...2/-

1. (a) Given a spline curve

$$R(u) = \begin{cases} P(u), & 0 \le u \le 1 \\ Q(u), & 1 \le u \le 3. \end{cases}$$

By parameter transformation, the curve segments $P(u)$ and $Q(u)$ can be represented locally as

$$P(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}(1-t)^2 + \begin{pmatrix} 0 \\ 2 \end{pmatrix}(1-t)t + \begin{pmatrix} a \\ b \end{pmatrix}t^2,$$

$$Q(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1-t)^3 + \begin{pmatrix} c \\ d \end{pmatrix}(1-t)^2 t + \begin{pmatrix} e \\ f \end{pmatrix}(1-t)t^2 + \begin{pmatrix} 3 \\ 0 \end{pmatrix}t^3,$$

where $t \in [0, 1]$, and $a, b, c, d, e, f \in \mathbb{R}$. If $R(u)$ is a tangent continuous (C^1) curve and at $u = 1$,

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}u^2}(1) = \frac{\mathrm{d}^2 P}{\mathrm{d}u^2}(1) + 2\frac{\mathrm{d}P}{\mathrm{d}u}(1),$$

evaluate the points (a, b), (c, d) and (e, f).

(b) Given a tensor product Bézier surface of degree $(2, 2)$

$$S(u, v) = \sum_{i=0}^{2}\sum_{j=0}^{2} C_{i,j}\, B_j^2(v) B_i^2(u), \qquad 0 \le u,\ v \le 1,$$

where B_k^2, $k = 0, 1, 2$, are the Bernstein polynomials of degree 2 and $C_{i,j} \in \mathbb{R}^3$ are the Bézier points.

(i) Show that $S(1, 0) = C_{2,0}$.

(ii) If

$$C_{0,0} = (0, 0, 2), \quad C_{1,0} = (1, 0, 1), \quad C_{2,0} = (2, 0, 2),$$
$$C_{0,1} = (0, 1, 1), \quad C_{1,1} = (1, 1, 0), \quad C_{2,1} = (2, 1, 1),$$
$$C_{0,2} = (0, 2, 2), \quad C_{1,2} = (1, 2, 1), \quad C_{2,2} = (2, 2, 2),$$

use the de Casteljau algorithm to evaluate $\frac{\partial^2}{\partial u\, \partial v} S(u,v)$ at $(u, v) = (0.3, 0.3)$.

[100 marks]

...3/-

1. *(a)* *Diberi suatu lengkung splin*

$$\boldsymbol{R}(u)=\begin{cases}\boldsymbol{P}(u), & 0\le u\le 1\\ \boldsymbol{Q}(u), & 1\le u\le 3.\end{cases}$$

Dengan menggunakan parameter transformasi, segmen lengkung $\boldsymbol{P}(u)$ *and* $\boldsymbol{Q}(u)$ *dapat diwakili secara setempat sebagai*

$$\boldsymbol{P}(t)=\begin{pmatrix}-1\\0\end{pmatrix}(1-t)^2+\begin{pmatrix}0\\2\end{pmatrix}(1-t)t+\begin{pmatrix}a\\b\end{pmatrix}t^2,$$

$$\boldsymbol{Q}(t)=\begin{pmatrix}1\\1\end{pmatrix}(1-t)^3+\begin{pmatrix}c\\d\end{pmatrix}(1-t)^2t+\begin{pmatrix}e\\f\end{pmatrix}(1-t)t^2+\begin{pmatrix}3\\0\end{pmatrix}t^3,$$

di mana $t\in[0,\ 1]$, *dan* $a,\ b,\ c,\ d,\ e,\ f\in\mathbb{R}$. *Jika* $\boldsymbol{R}(u)$ *ialah lengkung berkeselanjaran tangen* (C^1) *dan pada* $u=1$,

$$\frac{\mathrm{d}^2\boldsymbol{Q}}{\mathrm{d}u^2}(1)=\frac{\mathrm{d}^2\boldsymbol{P}}{\mathrm{d}u^2}(1)+2\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}u}(1),$$

nilaikan titik-titik $(a,\ b)$, $(c,\ d)$ *dan* $(e,\ f)$.

(b) *Diberi suatu permukaan hasil darab tensor Bézier berdarjah* $(2,\ 2)$

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{C}_{i,j}\,B_j^2(v)B_i^2(u),\qquad 0\le u,\ v\le 1,$$

di mana B_k^2, $k=0,\ 1,\ 2$, *ialah polinomial Bernstein berdarjah* 2 *dan* $\boldsymbol{C}_{i,j}\in\mathbb{R}^3$ *merupakan titik kawalan Bézier.*

(i) *Tunjukkan bahawa* $\boldsymbol{S}(1,\ 0)=\boldsymbol{C}_{2,0}$.

(ii) *Jika*

$$\boldsymbol{C}_{0,0}=(0,\ 0,\ 2),\quad \boldsymbol{C}_{1,0}=(1,\ 0,\ 1),\quad \boldsymbol{C}_{2,0}=(2,\ 0,\ 2),$$
$$\boldsymbol{C}_{0,1}=(0,\ 1,\ 1),\quad \boldsymbol{C}_{1,1}=(1,\ 1,\ 0),\quad \boldsymbol{C}_{2,1}=(2,\ 1,\ 1),$$
$$\boldsymbol{C}_{0,2}=(0,\ 2,\ 2),\quad \boldsymbol{C}_{1,2}=(1,\ 2,\ 1),\quad \boldsymbol{C}_{2,2}=(2,\ 2,\ 2),$$

gunakan algoritma de Casteljau untuk menilai $\dfrac{\partial^2}{\partial u\,\partial v}\boldsymbol{S}(u,v)$ *pada* $(u,\ v)=(0.3,\ 0.3)$.

[100 markah]

...4/-

2. (a) Given a knot vector $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n+k})$, $n \geq k-1$. Suppose a B-spline curve of order k is defined by

$$\boldsymbol{P}(u) = \sum_{i=0}^{n} \boldsymbol{D}_i N_i^k(u), \qquad u \in [u_{k-1}, u_{n+1}],$$

where $N_i^k(u)$ are the B-spline basis functions and $\boldsymbol{D}_i \in \mathbb{R}^2$ are the distinct de Boor points.

(i) Describe the conditions on the knot vector $\boldsymbol{u}$ so that the B-spline curve interpolates the first and the last de Boor points.

(ii) Prove that the curve segment $\boldsymbol{P}(u)$, $u \in [u_s, u_{s+1}]$, for $s = k-1, k, \ldots, n$, lies within the convex hull of the de Boor polygon $\boldsymbol{D}_{s-k+1}\boldsymbol{D}_{s-k+2}\ldots\boldsymbol{D}_s$.

(b) Let $N_i^4(u)$, $i = 0, 1, 2, 3$, be the B-spline basis functions of order 4 defined over the knot vector $\boldsymbol{u} = (0, 1, 2, 3, 4, 5, 6, 7)$. Given a B-spline curve

$$\boldsymbol{P}(u) = \begin{pmatrix}1\\0\end{pmatrix} N_0^4(u) + \begin{pmatrix}1\\1\end{pmatrix} N_1^4(u) + \begin{pmatrix}2\\1\end{pmatrix} N_2^4(u) + \begin{pmatrix}2\\0\end{pmatrix} N_3^4(u), \qquad u \in [3, 4].$$

(i) Find the control points of a cubic Bézier curve such that the curve coincides with $\boldsymbol{P}(u)$.

(ii) If a knot $u = 3.25$ is inserted into $\boldsymbol{u}$, find the de Boor points of a B-spline curve such that the curve coincides with $\boldsymbol{P}(u)$.

(c) Given knots $(u_i, v_j) = (i-2, j-2)$, $i = 0, 1, \ldots, 5$, $j = 0, 1, \ldots, 5$. A uniform tensor-product B-spline surface of order $(3, 3)$ is defined by

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \boldsymbol{D}_{i,j} N_j^3(v) N_i^3(u)$$

$$= \begin{bmatrix} u^2 & u & 1 \end{bmatrix} M \begin{bmatrix} \boldsymbol{D}_{0,0} & \boldsymbol{D}_{0,1} & \boldsymbol{D}_{0,2} \\ \boldsymbol{D}_{1,0} & \boldsymbol{D}_{1,1} & \boldsymbol{D}_{1,2} \\ \boldsymbol{D}_{2,0} & \boldsymbol{D}_{2,1} & \boldsymbol{D}_{2,2} \end{bmatrix} M^{\mathrm{T}} \begin{bmatrix} v^2 \\ v \\ 1 \end{bmatrix}, \qquad 0 \leq u,\ v \leq 1,$$

where N_s^3, $s = 0, 1, 2$, are the B-spline basis functions of order 3 and $\boldsymbol{D}_{i,j} \in \mathbb{R}^3$ are the de Boor points. Find the 3×3 matrix M.

[100 marks]

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2. (a) *Diberi vektor knot* $\boldsymbol{u} = (u_0,\ u_1,\ \ldots,\ u_{n+k})$, $n \geq k-1$. *Andaikan suatu splin-B berperingkat* k *ditakrifkan sebagai*

$$\boldsymbol{P}(u) = \sum_{i=0}^{n} \boldsymbol{D}_i\, N_i^k(u), \qquad u \in [u_{k-1},\ u_{n+1}],$$

di mana $N_i^k(u)$ *ialah fungsi asas splin-B dan* $\boldsymbol{D}_i \in \mathbb{R}^2$ *ialah titik-titik de Boor yang berbeza.*

(i) *Nyatakan syarat pada vektor knot* $\boldsymbol{u}$ *supaya lengkung splin-B menginterpolasi titik pertama dan titik akhir de Boor.*

(ii) *Buktikan bahawa segmen lengkung* $\boldsymbol{P}(u)$, $u \in [u_s,\ u_{s+1}]$, *for* $s = k-1,\ k,\ \ldots,\ n$, *terletak di dalam hul cembung bagi poligon de Boor* $\boldsymbol{D}_{s-k+1}\,\boldsymbol{D}_{s-k+2} \ldots \boldsymbol{D}_s$.

(b) *Katakan* $N_i^4(u)$, $i = 0,\ 1,\ 2,\ 3$, *merupakan fungsi-fungsi asas splin-B berperingkat* 4 *yang ditakrifkan pada vektor knot* $\boldsymbol{u} = (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7)$. *Diberi lengkung splin-B*

$$\boldsymbol{P}(u) = \begin{pmatrix}1\\0\end{pmatrix} N_0^4(u) + \begin{pmatrix}1\\1\end{pmatrix} N_1^4(u) + \begin{pmatrix}2\\1\end{pmatrix} N_2^4(u) + \begin{pmatrix}2\\0\end{pmatrix} N_3^4(u), \qquad u \in [3,\ 4].$$

(i) *Cari titik-titik kawalan bagi suatu lengkung Bézier kubik supaya lengkung ini sama dengan lengkung* $\boldsymbol{P}(u)$.

(ii) *Jika satu knot* $u = 3.25$ *dimasukkan ke dalam* $\boldsymbol{u}$, *cari titik-titik de Boor bagi suatu lengkung splin-B supaya lengkung ini sama dengan lengkung* $\boldsymbol{P}(u)$.

(c) *Diberi knot* $(u_i,\ v_j) = (i-2,\ j-2)$, $i = 0,\ 1,\ \ldots,\ 5$, $j = 0,\ 1,\ \ldots,\ 5$. *Permukaan hasil darab tensor splin-B seragam berperingkat* (3, 3) *ditakrifkan sebagai*

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \boldsymbol{D}_{i,j}\, N_j^3(v)\, N_i^3(u)$$

$$= \begin{bmatrix} u^2 & u & 1 \end{bmatrix} M \begin{bmatrix} \boldsymbol{D}_{0,0} & \boldsymbol{D}_{0,1} & \boldsymbol{D}_{0,2} \\ \boldsymbol{D}_{1,0} & \boldsymbol{D}_{1,1} & \boldsymbol{D}_{1,2} \\ \boldsymbol{D}_{2,0} & \boldsymbol{D}_{2,1} & \boldsymbol{D}_{2,2} \end{bmatrix} M^{\mathrm{T}} \begin{bmatrix} v^2 \\ v \\ 1 \end{bmatrix}, \qquad 0 \leq u,\ v \leq 1,$$

di mana N_s^3, $s = 0,\ 1,\ 2$, *ialah fungsi asas splin-B berperingkat* 3 *dan* $\boldsymbol{D}_{i,j} \in \mathbb{R}^3$ *ialah titik de Boor. Cari matriks* M *yang berperingkat* 3×3.

[100 markah]

...6/-

3. (a) Given a rational Bézier curve of the form

$$R(t) = \frac{C_0 B_0^2(t) + wC_1 B_1^2(t) + C_2 B_2^2(t)}{B_0^2(t) + wB_1^2(t) + B_2^2(t)}, \qquad t \in [0, 1],$$

where $B_i^2(t)$ are the Bernstein polynomials, $C_i \in \mathbb{R}^2$ are the Bézier points and the weight $w \geq 0$.

(i) Let $C_0 = (1, 0)$, $C_1 = (2, 1)$, $C_2 = (3, 0)$ and $w = 2$. Evaluate $\frac{d}{dt}R(t)$ at $t = 0.3$.

(ii) Let $C_0 = (1, 0)$, $C_1 = (1, q)$ and $C_2 = (a, b)$, where q, a, $b \in \mathbb{R}$. Evaluate the parameters w, q, a and b such that the curve $R(t)$ is the circular arc given as $R(\theta) = (\cos\theta, \sin\theta)$, $0 \leq \theta \leq \frac{3}{4}\pi$.

(b) Suppose a triangular Bézier surface of degree 2 is defined by

$$S(u, v, w) = \sum_{\substack{i+j+k=2 \\ i,j,k \geq 0}} C_{i,j,k} B_{i,j,k}^2(u, v, w), \qquad u + v + w = 1, \quad u, v, w \geq 0,$$

where $C_{i,j,k} \in \mathbb{R}^3$ are the Bézier points and

$$B_{i,j,k}^2(u, v, w) = \frac{2}{i!\ j!\ k!} u^i v^j w^k.$$

(i) Show that $S(u, v, w)$ at $v = 0$ is a quadratic Bézier curve with control points $C_{2,0,0}$, $C_{1,0,1}$ and $C_{0,0,2}$.

(ii) Let $C_{2,0,0} = (2, 2, 1)$, $C_{1,1,0} = (1, 1, 2)$, $C_{1,0,1} = (3, 1, 2)$, $C_{0,2,0} = (0, 0, 1)$, $C_{0,1,1} = (2, 0, 2)$ and $C_{0,0,2} = (4, 0, 1)$. If $S(u, v, w)$ is degree-elevated by one to

$$S(u, v, w) = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} C_{i,j,k}^* B_{i,j,k}^3(u, v, w),$$

evaluate the Bézier points $C_{i,j,k}^*$, for integers i, j, $k \geq 0$ and $i + j + k = 3$.

[100 marks]

3\. (a) *Diberi lengkung Bézier nisbah dalam bentuk*

$$\boldsymbol{R}(t)=\frac{\boldsymbol{C}_0 B_0^2(t)+w\boldsymbol{C}_1 B_1^2(t)+\boldsymbol{C}_2 B_2^2(t)}{B_0^2(t)+wB_1^2(t)+B_2^2(t)}, \qquad t\in[0,\ 1],$$

di mana $B_i^2(t)$ *ialah polinomial Bernstein,* $\boldsymbol{C}_i\in\mathbb{R}^2$ *ialah titik Bézier dan pemberat* $w\geq 0$.

(i) *Andaikan* $\boldsymbol{C}_0=(1,\ 0)$, $\boldsymbol{C}_1=(2,\ 1)$, $\boldsymbol{C}_2=(3,\ 0)$ *dan* $w=2$. *Nilaikan* $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}(t)$ *pada* $t=0.3$.

(ii) *Andaikan* $\boldsymbol{C}_0=(1,\ 0)$, $\boldsymbol{C}_1=(1,\ q)$ *dan* $\boldsymbol{C}_2=(a,\ b)$, *di mana* $q,\ a,\ b\in\mathbb{R}$. *Nilaikan parameter* w, q, a *and* b *supaya lengkung* $\boldsymbol{R}(t)$ *ialah lengkok bulatan yang ditakrifkan sebagai* $\boldsymbol{R}(\theta)=(\cos\theta,\ \sin\theta)$, $0\leq\theta\leq\frac{3}{4}\pi$.

(b) *Andaikan suatu permukaan segitiga Bézier berdarjah 2 ditakrifkan sebagai*

$$\boldsymbol{S}(u,\ v,\ w)=\sum_{\substack{i+j+k=2\\ i,\,j,\,k\geq 0}}\boldsymbol{C}_{i,j,k}\,B_{i,j,k}^2(u,\ v,\ w), \qquad u+v+w=1, \quad u,\ v,\ w\geq 0,$$

di mana $\boldsymbol{C}_{i,j,k}\in\mathbb{R}^3$ *ialah titik Bézier dan*

$$B_{i,j,k}^2(u,\ v,\ w)=\frac{2}{i!\ j!\ k!}u^i\,v^j\,w^k.$$

(i) *Tunjukkan bahawa* $\boldsymbol{S}(u,\ v,\ w)$ *pada* $v=0$ *merupakan suatu lengkung Bézier kuadratik dengan titik-titik kawalan* $\boldsymbol{C}_{2,0,0}$, $\boldsymbol{C}_{1,0,1}$ *dan* $\boldsymbol{C}_{0,0,2}$.

(ii) *Katakan* $\boldsymbol{C}_{2,0,0}=(2,\ 2,\ 1)$, $\boldsymbol{C}_{1,1,0}=(1,\ 1,\ 2)$, $\boldsymbol{C}_{1,0,1}=(3,\ 1,\ 2)$, $\boldsymbol{C}_{0,2,0}=(0,\ 0,\ 1)$, $\boldsymbol{C}_{0,1,1}=(2,\ 0,\ 2)$ *dan* $\boldsymbol{C}_{0,0,2}=(4,\ 0,\ 1)$. *Jika* $\boldsymbol{S}(u,\ v,\ w)$ *ditingkatkan darjahnya dengan satu kepada*

$$\boldsymbol{S}(u,\ v,\ w)=\sum_{\substack{i+j+k=3\\ i,\,j,\,k\geq 0}}\boldsymbol{C}^*_{i,j,k}\,B_{i,j,k}^3(u,\ v,\ w),$$

kira titik-titik Bézier $\boldsymbol{C}^*_{i,j,k}$, *bagi integer* $i,\ j,\ k\geq 0$ *dan* $i+j+k=3$.

[100 markah]

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