UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

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1. (a) Given $OAPB$ is a parallelogram and Q is the midpoint of AP. OP meets BQ at X.

   (i) If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$, find $\overrightarrow{BQ}$ and $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (ii) If R is a point on the line extended from BQ such that $\overrightarrow{BQ} = \overrightarrow{QR}$ then show that R is also on the line extended from OA and that $\overrightarrow{OR} = 2\overrightarrow{OA}$.

   (iii) Let $\overrightarrow{OX} = r\overrightarrow{OP}$ and $\overrightarrow{BX} = s\overrightarrow{BQ}$. By expressing $\overrightarrow{OX}$ in two different ways, find the values of r and s.

   (iv) Find the ratios $\dfrac{OX}{XP}$ and $\dfrac{BX}{XQ}$.

   (b) In $\mathbb{R}^3$, let L be the line through points $P(5,0,3)$ and $Q(6,5,2)$.

   (i) Find equations, in both vector form and parametric form of the line L.

   (ii) Find the point of intersection of the line L and the plane with equation $x + y + z + 2 = 0$.

   (c) Find an equation of the plane containing the points $P(1,1,1), Q(2,2,0)$ and $R(3,0,0)$. Express your answer in the general form $ax + by + cz + d = 0$.

   (d) (i) A parallelogram in $\mathbb{R}^3$ has as adjacent sides the vectors $\boldsymbol{u} = (1,3,-2)$ and $\boldsymbol{v} = (3,-1,-1)$. Determine the area of the parallelogram.

   (ii) Find the volume of the parallelepiped with a vertex at the origin and edges $\boldsymbol{u} = 2\vec{i} - \vec{j}, \boldsymbol{v} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\boldsymbol{w} = 3\vec{i} - 2\vec{j} + \vec{k}$.

[100 marks]

2. Let $W = \left\{(x,y,z) \in \mathbb{R}^3 \middle| x - y + 3z = 0\right\}$

   (a) Show that W is a subspace of $\mathbb{R}^3$.

   (b) (i) Find the set S such that $L(S) = W$.

   (ii) Deduce that S is a basis of W.

   (c) (i) Show that the vector $(4,7,1)$ is in W.

   (ii) Add this vector to S to form $S' = S \cup \{(4,7,1)\}$

   (iii) Explain why S' is not a basis.

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1. (a) *Diberi OAPB adalah suatu segiempat selari dan Q ialah titik tengah AP. OP bertemu BQ pada X.*

   (i) *Jika* $\overrightarrow{OA} = \boldsymbol{a}$ *dan* $\overrightarrow{OB} = \boldsymbol{b}$, *cari* $\overrightarrow{BQ}$ *dan* $\overrightarrow{OP}$ *dalam sebutan* **a** *dan* **b**.

   (ii) *Jika R ialah suatu titik pada garislurus yang dilanjutkan dari BQ sedemikian hingga* $\overrightarrow{BQ} = \overrightarrow{QR}$ *maka tunjukkan bahawa R juga ialah suatu titik pada garislurus yang dilanjutkan dari OA dengan* $\overrightarrow{OR} = 2\overrightarrow{OA}$.

   (iii) *Biar* $\overrightarrow{OX} = r\,\overrightarrow{OP}$ *dan* $\overrightarrow{BX} = s\,\overrightarrow{BQ}$. *Dengan mengungkapkan* $\overrightarrow{OX}$ *dalam dua cara, cari nilai-nilai r dan s.*

   (iv) *Cari nisbah* $\dfrac{OX}{XP}$ *dan* $\dfrac{BX}{XQ}$.

   (b) *Dalam* $\mathbb{R}^3$, *biar L sebagai satu garislurus yang melalui titik-titik* $P(5,0,3)$ *dan* $Q(6,5,2)$.

   (i) *Cari persamaan-persamaan, dalam kedua-dua bentuk vektor dan bentuk berparameter, bagi garislurus L tersebut.*

   (ii) *Cari titik persilangan L dengan satah yang mempunyai persamaan* $x + y + z + 2 = 0$.

   (c) *Cari persamaan satah yang mengandungi titik-titik* $P(1,1,1), Q(2,2,0)$ *dan* $R(3,0,0)$. *Ungkapkan jawapan anda dalam bentuk umum* $ax + by + cz + d = 0$.

   (d) (i) *Suatu segiempat selari dalam* $\mathbb{R}^3$ *mempunyai vektor-vektor* $\boldsymbol{u} = (1,3,-2)$ *dan* $\boldsymbol{v} = (3,-1,-1)$ *sebagai sisi-sisinya. Tentukan luas segiempat selari tersebut.*

   (ii) *Cari isipadu paralelepiped dengan satu bucu pada asalan dan sisi-sisi* $\boldsymbol{u} = 2\vec{i} - \vec{j}, \boldsymbol{v} = \vec{i} - 2\vec{j} - 2\vec{k}$ *dan* $\boldsymbol{w} = 3\vec{i} - 2\vec{j} + \vec{k}$.

[100 markah]

2. *Biar* $W = \left\{(x,y,z) \in \mathbb{R}^3 \middle| x - y + 3z = 0\right\}$

   (a) *Tunjukkan bahawa W ialah subruang* $\mathbb{R}^3$.

   (b) (i) *Cari set S sedemikian hingga* $L(S) = W$.

   (ii) *Deduksikan bahawa S adalah asas W.*

   (c) (i) *Tunjukkan bahawa vektor* $(4,7,1)$ *berada dalam W.*

   (ii) *Tambahkan vektor ini kepada S untuk membentuk* $S' = S \cup \{(4,7,1)\}$

   (iii) *Terangkan mengapa* S' *bukan suatu asas.*

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(d) (i) Show that the vector $(1,0,0)$ is not in W.

(ii) Add this vector to S to form $S'' = S \cup \{(1,0,0)\}$.

(iii) Explain why S'' is a basis.

(iv) What is $L(S'')$ and the dimension of $L(S'')$?

[100 marks]

3. (a) Find the basis and dimension of

$$V = \{f(x) \in P_2(\mathbb{R}) \ | f(0) = 0, f(1) = f(2)\}$$

(b) Let T be a function such that $T: P_2(\mathbb{R}) \to M_{2\times 2}$ and defined by

$$(f(x))T = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$$

(i) Show that T is a linear transformation.

(ii) Given that $\{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$.

Find $\operatorname{Im} T$. What is $\dim(\operatorname{Im} T)$?

[Hint: Find $(1)T$, $(x)T$ and $(x^2)T$ and use them to find $(f(x))T$]

(iii) By the Dimension Theorem, what is $\dim(\operatorname{Ker} T)$?

(iv) Verify your answer in (iii) by finding the basis or $\operatorname{Ker} T$.

(c) Let $T: V \to W$ be a linear transformation between two finite-dimensional vector spaces.

(i) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(ii) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

(d) Given that $I: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined by $(v)I = v$. Let α be the standard ordered basis of $\mathbb{R}^3$ and $\beta = \{(1,1,1), (1,1,0)(1,0,0)\}$.

(i) Show that β is also an ordered basis of $\mathbb{R}^3$.

(ii) Determine $I_{\alpha,\beta}$ and $I_{\beta,\alpha}$.

(iii) Find $(3,2,1)_\alpha$ and $(3,2,1)_\beta$? How are these related?

[100 marks]

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(d) (i) *Tunjukkan bahawa vektor* $(1,0,0)$ *bukan dalam W.*

(ii) *Tambahkan vektor ini kepada S untuk membentuk* $S'' = S \cup \{(1,0,0)\}$.

(iii) *Terangkan mengapa* S'' *ialah suatu asas.*

(iv) *Apakah* $L(S'')$ *dan dimensi* $L(S'')$ *?*

[100 markah]

3. *(a) Cari asas dan dimensi bagi*

$$V = \{f(x) \in P_2(\mathbb{R}) \mid f(0) = 0, f(1) = f(2)\}$$

(b) Biar T sebagai fungsi sedemikian hingga $T : P_2(\mathbb{R}) \to M_{2\times 2}$ *dan ditakrifkan dengan*

$$(f(x))T = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$$

(i) *Tunjukkan bahawa T ialah suatu transformasi linear.*

(ii) *Diberi bahawa* $\{1, x, x^2\}$ *ialah asas* $P_2(\mathbb{R})$.

Cari $\operatorname{Im} T$. *Apakah* $\dim(\operatorname{Im} T)$ *?*

(iii) *Dari Teorem Dimensi, apakah* $\dim(\operatorname{Ker} T)$ *?*

(iv) *Tahkikkan jawapan anda dalam (iii) dengan mencari asas* $\operatorname{Ker} T$.

(c) Biar $T : V \to W$ *sebagai suatu transformasi linear di antara dua ruang vektor berdimensi terhingga.*

(i) *Buktikan bahawa jika* $\dim(V) < \dim(W)$, *maka T bukan keseluruhan.*

(ii) *Buktikan bahawa jika* $\dim(V) > \dim(W)$, *maka T bukan satu-ke-satu.*

(d) Diberi $I : \mathbb{R}^3 \to \mathbb{R}^3$ *ialah suatu transformasi linear yang ditakrifkan dengan* $(v)I = v$. *Biar* α *suatu asas piawai bertertib bagi* $\mathbb{R}^3$ *dan* $\beta = \{(1,1,1), (1,1,0)\,(1,0,0)\}$.

(i) *Tunjukkan bahawa* β *juga suatu asas bertertib bagi* $\mathbb{R}^3$.

(ii) *Tentukan* $I_{\alpha,\beta}$ *dan* $I_{\beta,\alpha}$.

(iii) *Cari* $(3,2,1)_\alpha$ *dan* $(3,2,1)_\beta$ *? Apakah hubungan mereka?*

[100 markah]

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4. (a) Let W be a subspace of $\mathbb{R}^2$ such that $W = \{(x,y) \mid 3x+4y=0\}$.

    (i) Find the orthogonal complement, $W^{\perp}$, of W.

    (ii) Give the basis of $W^{\perp}$.

(b) Use the Gram-Schmidt process to find an orthonormal basis from the set $\{(2,0,0),(1,1,1),(0,1,0)\}$

(c) Find the best approximate solution to the following overdetermined system:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 2 \\ 3x_1 + 5x_2 &= 0 \end{aligned}$$

(d) Given

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ -1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -2 & 2 & -3 & 1 & 0 \end{bmatrix}$$

    (i) What is the column space of A?

    (ii) Find the basis and dimension of the column space of A and deduce the rank of A.

[100 marks]

5. Given the following system of linear equations

$$\begin{array}{rcrcrcrcl} x & & & + & 2z & - & w & = & 0 \\ -x & + & y & & -z & + & 2w & = & 1 \\ & & y & + & z & + & w & = & 1 \\ -2x & + & y & - & 3z & + & 2w & = & 0 \end{array}$$

(a) Write out the matrix A where $A\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ represent the above system. Then with the matrix A, write out the augmented matrix $[A|b]$ where $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

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4. (a) *Biar W sebagai suatu subruang dari* $\mathbb{R}^2$ *dengan* $W = \{(x,y) \mid 3x+4y=0\}$.

   (i) *Cari pelengkap berortogon,* $W^{\perp}$, *bagi W.*

   (ii) *Berikan asas* $W^{\perp}$.

   (b) *Gunakan proses Gram-Schmidt untuk mencari asas berortonormal dari set* $\{(2,0,0),(1,1,1),(0,1,0)\}$

   (c) *Cari penyelesaian hampiran terbaik bagi sistem lebihtertentu berikut:*

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 2 \\ 3x_1 + 5x_2 &= 0 \end{aligned}$$

   (d) *Diberi*

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ -1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -2 & 2 & -3 & 1 & 0 \end{bmatrix}$$

   (i) *Cari ruang lajur A.*

   (ii) *Cari asas dan dimensi ruang lajur A dan deduksikan pangkat A.*

[100 markah]

5. *Diberi sistem persamaan linear berikut*

$$\begin{array}{rcrcrcrcr} x & & & + & 2z & - & w & = & 0 \\ -x & + & y & & -z & + & 2w & = & 1 \\ & & y & + & z & + & w & = & 1 \\ -2x & + & y & - & 3z & + & 2w & = & 0 \end{array}$$

   (a) *Tuliskan matriks A yang mana* $A\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ *mewakili sistem di atas. Kemudian dengan matriks A yang diperoleh dari (a), tuliskan matriks imbuhan* $[A|b]$ *dengan* $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

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(b) Use the Gauss-Jordan process to obtain the row-reduced echelon form of $[A|b]$.

(c) Write out the equation obtained from the row-reduced echelon form in (b). Determine the leading and free variables and then solve the system to get the solution in the form

$$(x, y, z, w) = (a, b, c, d) + \alpha(a', b', c', d'), \alpha \in \mathbb{R}.$$

(d) Check your solution by showing that (a, b, c, d) is a particular solution to the system and $\alpha(a', b', c', d') \in \operatorname{Ker} A$.

[100 marks]

(b) *Gunakan proses Gauss-Jordan untuk mendapatkan bentuk eselon baris terturun* $[A|b]$.

(c) *Tuliskan persamaan-persamaan yang terhasil daripada bentuk eselon baris terturun dalam (b). Tentukan pembolehubah-pembolehubah utama dan bebas dan kemudian selesaikan sistem tersebut untuk mendapatkan penyelesaian dalam bentuk*

$$(x,y,z,w)=(a,b,c,d)+\alpha(a',b',c',d'),\ \alpha\in\mathbb{R}.$$

(d) *Semak jawapan anda dengan menunjukkan bahawa* (a,b,c,d) *ialah suatu penyelesaian tertentu bagi sistem tersebut dan* $\alpha(a',b',c',d')\in \operatorname{Ker} A$.

[100 markah]

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