UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 101 – Calculus
[Kalkulus]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Find the following limits:

(i) $\lim_{x\to 4} \dfrac{x-\sqrt{5x-4}}{4-x}$

(ii) $\lim_{x\to 0} \dfrac{\tan^{-1} 2x}{x}$

(iii) $\lim_{x\to -2} \left((x^2-4)^3 \cos\dfrac{\pi}{x+2} \right)$

(b) Prove by using the definition of limit that $\lim_{x\to 2} \dfrac{6}{2+x^2} = 1$.

(c) (i) Prove that $\max\{a,b\} = \dfrac{a+b}{2} + \left|\dfrac{a-b}{2}\right|$, $\forall\, a,b \in \mathbb{R}$.

(ii) Write the similar expression for $\min\{a,b\}$, $\forall\, a,b \in \mathbb{R}$.
(You do not need to prove it)

(d) If $f,\ g : [a,b] \to [a,b]$ are continuous functions with $f(a)=a$, $f(b)=b$, $g(a)=b$, $g(b)=a$, show that $\exists\ c \in (a,b)$ such that $f(c)=g(c)$.

[100 Marks]

2. (a) Given that $f(x) = \begin{cases} 2x, & x<1 \\ \sqrt{x+3}, & x \ge 1 \end{cases}$.

Show that f is continuous at $x=1$ but not differentiable at $x=1$.

(b) Given that $f(x) = \dfrac{9x}{(x+3)^2}$, $x \in \mathbb{R}$ and $x \neq -3$.

(i) Determine where the graph of f is increasing or decreasing and find the local extremum of f.

(ii) Determine where the graph of f is convex or concave and find the inflection point of f.

(iii) Find all the asymptotes of f.

(iv) With the above properties of f, sketch the graph of f.

1. (a) *Cari had yang berikut:*

   (i) $\lim_{x \to 4} \dfrac{x - \sqrt{5x-4}}{4-x}$

   (ii) $\lim_{x \to 0} \dfrac{\tan^{-1} 2x}{x}$

   (iii) $\lim_{x \to -2} \left((x^2-4)^3 \cos \dfrac{\pi}{x+2} \right)$

   (b) *Buktikan dengan menggunakan takrif had bahawa* $\lim_{x \to 2} \dfrac{6}{2+x^2} = 1$.

   (c) (i) *Buktikan bahawa* $\max\{a, b\} = \dfrac{a+b}{2} + \left| \dfrac{a-b}{2} \right|, \quad \forall\, a, b \in \mathbb{R}$.

   (ii) *Tuliskan ungkapan yang serupa bagi* $\min\{a, b\}, \ \forall\, a, b \in \mathbb{R}$.
   (Anda tidak perlu buktikannya)

   (d) *Jika* $f,\ g : [a, b] \to [a, b]$ *adalah fungsi yang selanjar dengan* $f(a) = a$, $f(b) = b$, $g(a) = b$, $g(b) = a$, *tunjukkan bahawa* $\exists\ c \in (a, b)$ *supaya* $f(c) = g(c)$.

[100 Markah]

2. (a) *Diberi* $f(x) = \begin{cases} 2x, & x < 1 \\ \sqrt{x+3}, & x \geq 1 \end{cases}$.

   Tunjukkan bahawa f *adalah selanjar pada* $x = 1$ *tetapi tidak terbezakan pada* $x = 1$.

   (b) *Diberi* $f(x) = \dfrac{9x}{(x+3)^2}$, $x \in \mathbb{R}$ *dan* $x \neq -3$.

   (i) *Tentukan di mana graf bagi* f *adalah menokok atau menyusut dan cari ekstremum setempat bagi* f.

   (ii) *Tentukan di mana graf bagi* f *adalah cembung atau cekung dan cari titik lengkok balas bagi* f.

   (iii) *Cari semua asimptot bagi* f.

   (iv) *Dengan sifat-sifat* f *di atas, lakarkan graf bagi* f.

(c) (i) State the Mean Value Theorem.

(ii) By using the Mean Value Theorem, prove that if $f'(x) = 0,\ \forall\, x \in \mathbb{R}$, then f is a constant function (i.e. $f(a) = f(b),\ \forall\, a, b \in \mathbb{R}$).

(iii) With the help of (ii) above, if $h'(x) = g'(x),\ \forall\, x \in \mathbb{R}$, prove that

$$h(x) - g(x) = K, \quad \forall\, x \in \mathbb{R},$$

where K is some fixed number.

Hence if $g(x) = \sqrt{3x^2 + 6},\ \forall\, x \in \mathbb{R}$, find the function h such that

$$h'(x) = g'(x),\ \forall\, x \in \mathbb{R} \quad \text{and} \quad h(-5) = 7.$$

[100 Marks]

3. (a) Differentiate the following functions:

(i) $\tan^{-1}\sqrt{4 + e^{x^2}}$

(ii) $\dfrac{(5x^4+1)^{\sqrt{x}} \sin 2x}{\sqrt{7+3x^2}}$

(b) If $F(x) = \displaystyle\int_1^{x^2} \frac{\ln t}{1 + (\ln t)^5}\, dt$, find $F'(x)$ and show that $F'(e) = \dfrac{4e}{33}$.

(c) (i) Given that the function $g(x) = x - \tan x,\ x \in \left[0, \frac{\pi}{2}\right)$.

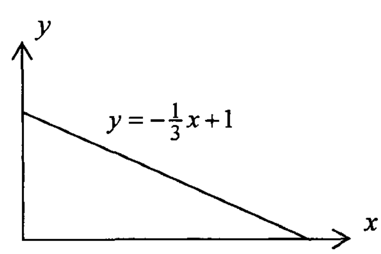
Determine whether g is decreasing on $\left[0, \frac{\pi}{2}\right)$.

Hence show that $x - \tan x \le 0$ for $x \in \left[0, \frac{\pi}{2}\right)$.

(ii) Show that $\dfrac{\sin x}{x}$ is decreasing on the interval $\left(0, \frac{\pi}{2}\right]$.

Hence show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx \le \frac{1}{\sqrt{2}}$.

(d) A triangle is bounded by the x-axis, y-axis and the straight line $y = -\frac{1}{3}x + 1$ as shown in the figure. Find the area of the largest rectangle that lies inside the triangle with two of its sides along the x-axis and y-axis respectively and one of its vertices on the line $y = -\frac{1}{3}x + 1$.



[100 Marks]

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(c) (i) *Nyatakan Teorem Nilai Min.*

(ii) *Dengan menggunakan Teorem Nilai Min, buktikan bahawa jika* $f'(x) = 0$, $\forall\, x \in \mathbb{R}$, *maka* f *ialah suatu fungsi malar (iaitu* $f(a) = f(b)$, $\forall\, a, b \in \mathbb{R}$*).*

(iii) *Dengan bantuan dari (ii) di atas, jika* $h'(x) = g'(x)$, $\forall\, x \in \mathbb{R}$, *buktikan bahawa*

$$h(x) - g(x) = K, \quad \forall\, x \in \mathbb{R},$$

di mana K *ialah suatu nombor yang tetap.*

Dengan itu, jika $g(x) = \sqrt{3x^2 + 6}$, $\forall\, x \in \mathbb{R}$, *cari fungsi* h *supaya*

$$h'(x) = g'(x),\ \forall\, x \in \mathbb{R} \quad \text{dan} \quad h(-5) = 7.$$

[100 Markah]

3. (a) *Bezakan fungsi yang berikut:*

(i) $\tan^{-1}\sqrt{4 + e^{x^2}}$

(ii) $\dfrac{(5x^4 + 1)^{\sqrt{x}} \sin 2x}{\sqrt{7 + 3x^2}}$

(b) *Jika* $F(x) = \displaystyle\int_1^{x^2} \frac{\ln t}{1 + (\ln t)^5}\, dt$, *cari* $F'(x)$ *dan tunjukkan* $F'(e) = \dfrac{4e}{33}$.

(c) (i) *Diberi fungsi* $g(x) = x - \tan x$, $x \in \left[0, \frac{\pi}{2}\right)$.

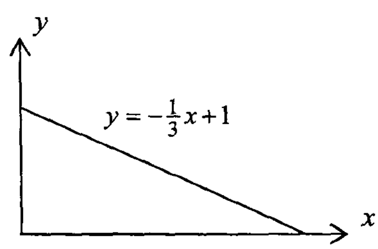
Tentukan sama ada g *adalah menyusut pada* $\left[0, \frac{\pi}{2}\right)$.

Dengan itu, tunjukkan bahawa $x - \tan x \le 0$ *bagi* $x \in \left[0, \frac{\pi}{2}\right)$.

(ii) *Tunjukkan bahawa* $\dfrac{\sin x}{x}$ *adalah menyusut pada* $\left(0, \frac{\pi}{2}\right]$.

Dengan itu, tunjukkan bahawa $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx \le \frac{1}{\sqrt{2}}$.

(d) *Suatu segitiga dibatasi oleh paksi* x, *paksi* y *dan garis lurus* $y = -\frac{1}{3}x + 1$ *seperti yang diitunjuk dalam rajah. Cari luas segiempat tepat yang terbesar yang terletak dalam segitiga dengan dua sisinya di sepanjang paksi* x *dan paksi* y *masing-masing dan suatu bucunya terletak pada garis* $y = -\frac{1}{3}x + 1$.



[100 Markah]

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4. (a) Find the following integrals:

   (i) $\int x^2 \tan^{-1} x \, dx$

   (ii) $\int \frac{\cos x}{\sin^2 x - \sin x - 2} \, dx$

   (iii) $\int \frac{\sqrt{x^2 - 4}}{x^4} \, dx$

(b) The region D in the first quadrant is bounded by the x-axis, y-axis and the circular arc from $x^2 + y^2 = a^2$ where $a > 0$. Find the volume of the hemispherical solid formed by revolving D about the y-axis.

(c) Suppose that the function $f : [0,1] \to \mathbb{R}$ is defined as $f(x) = \frac{1}{1+x}$ and the partition P_n on $[0,1]$ is defined as $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$.

   (i) Show that the upper sum $U(P_n, f)$ and lower sum $L(P_n, f)$ of f with respect to the partition P_n are given by

   $$U(P_n, f) = \sum_{k=1}^{n} \frac{1}{n+k-1}$$

   and

   $$L(P_n, f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

   (ii) Hence show that f is integrable on $[0,1]$ by using the Riemann criterion.

[100 Marks]

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4. (a) *Cari kamiran yang berikut:*

(i) $\int x^2 \tan^{-1} x \, dx$

(ii) $\int \dfrac{\cos x}{\sin^2 x - \sin x - 2} \, dx$

(iii) $\int \dfrac{\sqrt{x^2 - 4}}{x^4} \, dx$

(b) *Rantau* D *di sukuan pertama dibatasi oleh paksi* x, *paksi* y *dan lengkok dari bulatan* $x^2 + y^2 = a^2$ *di mana* $a > 0$. *Cari isipadu bongkah hemisfera yang terbentuk dengan mengisar* D *terhadap paksi* y.

(c) *Andaikan fungsi* $f : [0,1] \to \mathbb{R}$ *ditakrifkan sebagai* $f(x) = \dfrac{1}{1+x}$ *dan partisi* P_n *pada* $[0,1]$ *ditakrifkan sebagai* $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$.

(i) *Tunjukkan bahawa hasil tambah atas* $U(P_n, f)$ *dan hasil tambah bawah* $L(P_n, f)$ *bagi* f *terhadap partisi* P_n *diberikan sebagai*

$$U(P_n, f) = \sum_{k=1}^{n} \frac{1}{n+k-1}$$

dan

$$L(P_n, f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

(ii) *Dengan itu, tunjukkan bahawa* f *adalah terkamirkan pada* $[0,1]$ *melalui kriterior Riemann.*

[100 Markah]

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