UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2007/2008

October/November 2007

MAT 263 – Teori Kebarangkalian
[Probability Theory]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all five** [5] questions.

*[**Arahan:** Jawab **semua lima** [5] soalan.]*

1. (a) Suppose that the distribution function of random variable X is given by

$$F(x)=\begin{cases} 0 & , \quad x<1 \\ \frac{1}{8} & , \quad 1\le x<2 \\ \frac{3}{8} & , \quad 2\le x<3 \\ \frac{3}{4} & , \quad 3\le x<4 \\ 1 & , \quad x\ge 4. \end{cases}$$

(i) Find $P(X=2)$, $P(X<3)$, $P(X>2)$ and $P(1\le X<3)$.

(ii) Determine the probability density function (p.d.f.) of X.

[40 marks]

(b) Suppose that a random variable X is uniformly distributed over the interval (50, 100).

(i) Find the mean and variance of X.

(ii) By using the Chebyshev's Inequality, determine the upper bound of $P(|X-25|\ge 20)$.

(iii) Compare the upper bound in (ii) with the exact probability $P(|X-25|\ge 20)$.

[40 marks]

(c) If two events A and B are independent, show that the events A' and B' are also independent.

[20 marks]

2. (a) A random variable X has p.d.f. defined by

$$f(x)=\lambda x^{\lambda-1} \quad , \quad 0<x<1, \quad \lambda>0.$$

Find the p.d.f. of the random variable Y where $Y=-2\lambda \ln X$. What is the distribution of Y?

[30 marks]

(b) A company produces a certain type of transistor. The probability that the transistors will be defective is 0.05, independently of each other. The company sells the transistors in packages of size 12 and offers a money-back guarantee that at most 2 of the 12 transistors in the packages will be defective. If you buy 5 packages, what is the probability that none of the packages will be returned?

[30 marks]

...3/-

1. (a) *Andaikan fungsi taburan bagi pembolehubah rawak X diberi sebagai*

$$F(x)=\begin{cases} 0 & , \quad x<1 \\ \frac{1}{8} & , \quad 1\le x<2 \\ \frac{3}{8} & , \quad 2\le x<3 \\ \frac{3}{4} & , \quad 3\le x<4 \\ 1 & , \quad x\ge 4. \end{cases}$$

(i) *Cari* $P(X=2)$, $P(X<3)$, $P(X>2)$ *dan* $P(1\le X<3)$.

(ii) *Tentukan fungsi ketumpatan kebarangkalian (f.k.k.) bagi X.*

[40 markah]

(b) *Andaikan pembolehubah rawak X mempunyai taburan seragam dalam selang (50, 100).*

(i) *Dapatkan min dan varians bagi X.*

(ii) *Dengan menggunakan ketaksamaan Chebyshev, tentukan batas atas bagi* $P(|X-25|\ge 20)$.

(iii) *Bandingkan batas atas dalam bahagian (ii) dengan kebarangkalian tepat* $P(|X-25|\ge 20)$.

[40 markah]

(c) *Jika dua peristiwa A dan B tidak bersandar, tunjukkan bahawa peristiwa* A' *dan peristiwa* B' *juga adalah tidak bersandar.*

[20 markah]

2. (a) *Pembolehubah rawak X mempunyai f.k.k. yang ditakrif sebagai*

$$f(x)=\lambda x^{\lambda-1} \quad , \quad 0<x<1, \quad \lambda>0.$$

Dapatkan f.k.k. bagi pembolehubah rawak Y yang mana $Y=-2\lambda\ln X$. *Apakah taburan Y?*

[30 markah]

(b) *Sebuah syarikat mengeluarkan sejenis transistor. Kebarangkalian bahawa transistor-transistor itu rosak adalah 0.05, tidak bersandar antara satu sama lain. Syarikat tersebut menjual transistor-transistor ini dalam pakej bersaiz 12 dan menawarkan jaminan pengembalian wang bahawa paling banyak 2 daripada 12 transistor itu rosak. Jika anda membeli sebanyak 5 pakej, apakah kebarangkalian yang tidak satu pun daripada 5 pakej itu akan dikembalikan?*

[30 markah]

...4/-

(c) X and Y are two independent random variables with joint p.d.f. given by

$$f(x,y) = \begin{cases} \dfrac{x^{\alpha-1}\, y^{\beta-1}\, e^{-(x+y)}}{\Gamma(\alpha)\,\Gamma(\beta)} , & x>0, \quad y>0 \\ & \alpha, \beta > 0 \\ 0 , & \text{elsewhere.} \end{cases}$$

(i) Find the joint p.d.f. of random variables U and V where

$$U = \frac{X}{X+Y} \quad \text{and} \quad V = X+Y.$$

(ii) Find the marginal p.d.f. of U and of V.

(iii) Are U and V independent? Why?

[40 marks]

3. (a) Suppose that the joint p.d.f. of random variables X and Y is given by

$$f(x,y) = \frac{2x+y}{27}, \quad x=0,1,2, \quad y=2,3.$$

(i) Verify that $f(x,y)$ is a joint p.d.f.

(ii) Find the marginal p.d.f. of X.

(iii) Compute $P(XY>3)$, $P(Y-X<2)$ and $P(Y=2|X=2)$.

[30 marks]

(b) Let the random variables X and Y have the following joint p.d.f.:

$$f(x,y) = \begin{cases} c , & 0 \le y \le x^2 \le 1 \\ 0 , & \text{elsewhere.} \end{cases}$$

(i) Find the constant c.

(ii) Find the marginal p.d.f. of Y.

(iii) Find the conditional p.d.f. of X given $Y = \frac{1}{4}$.

(iv) Determine $E\left(X^2 \middle| Y = \frac{1}{4}\right)$.

[50 marks]

(c) Suppose that X and Y are random variables with $\sigma_X^2 > 0$ and $\sigma_X^2 > 0$. If $Y = a + bX$, $a \neq 0$ and $b \neq 0$, show that $\rho_{XY} = 1$ if $b > 0$ and $\rho_{XY} = -1$ if $b < 0$.

[20 marks]

...5/-

(c) *X dan Y ialah dua pembolehubah rawak dengan f.k.k. tercantum diberi oleh*

$$f(x,y)=\begin{cases}\dfrac{x^{\alpha-1}\,y^{\beta-1}\,e^{-(x+y)}}{\Gamma(\alpha)\,\Gamma(\beta)} & , \quad x>0,\quad y>0 \\ & \quad\ \ \alpha,\beta>0 \\ 0 & , \quad \text{di tempat lain.}\end{cases}$$

(i) *Dapatkan f.k.k. tercantum bagi pembolehubah-pembolehubah rawak U dan V yang mana*

$$U=\frac{X}{X+Y} \qquad \text{dan} \qquad V=X+Y.$$

(ii) *Dapatkan f.k.k. sut bagi U dan f.k.k. sut bagi V.*

(iii) *Adakah U dan V tak bersandar? Kenapa?*

[40 markah]

3. (a) *Andaikan f.k.k. tercantum bagi pembolehubah-pembolehubah rawak X dan Y diberi oleh*

$$f(x,y)=\frac{2x+y}{27}, \quad x=0,1,2, \quad y=2,3.$$

(i) *Sahkan bahawa $f(x,y)$ adalah suatu f.k.k. tercantum.*

(ii) *Dapatkan f.k.k. sut bagi X.*

(iii) *Hitung $P(XY>3)$, $P(Y-X<2)$ dan $P(Y=2|X=2)$.*

[30 markah]

(b) *Biarkan pembolehubah-pembolehubah rawak X dan Y mempunyai f.k.k. tercantum berikut:*

$$f(x,y)=\begin{cases} c & , \quad 0\le y\le x^2\le 1 \\ 0 & , \quad \text{di tempat lain.}\end{cases}$$

(i) *Dapatkan pemalar c.*

(ii) *Dapatkan f.k.k. sut bagi Y.*

(iii) *Dapatkan f.k.k. bersyarat bagi X diberi $Y=\frac{1}{4}$.*

(iv) *Tentukan $E\left(X^2 \middle| Y=\frac{1}{4}\right)$.*

[50 markah]

(c) *Andaikan bahawa X dan Y ialah pembolehubah-pembolehubah rawak dengan $\sigma_X^2>0$ dan $\sigma_X^2>0$. Jika $Y=a+bX$, $a\neq 0$ dan $b\neq 0$, tunjukkan bahawa $\rho_{XY}=1$ jika $b>0$ dan $\rho_{XY}=-1$ jika $b<0$.*

[20 markah]

...6/-

4. (a) (i) The moment generating function (m.g.f.) of random variable X is given by

$$M_X(t) = \frac{1}{3} + \frac{1}{4}e^t + \frac{1}{6}e^{3t} + \frac{1}{4}e^{5t}.$$

Find $P(0 < X \leq 4)$.

(ii) The m.g.f. of random variable X is given by

$$M_X(t) = e^{3(e^t - 1)},$$

and that of random variable Y by

$$M_Y(t) = (0.3e^t + 0.7)^4.$$

Find $E(2X + 3Y + 1)$.

[30 marks]

(b) An urn contains 3 white and 2 blue marbles. The marbles are to be chosen, one at a time, until these two blue marbles are obtained. Denote N as the number of trials required until the first blue marble is selected and M as the number of additional trials until the second blue marble is selected. Find the joint p.d.f. of N and M.

[30 marks]

(c) Mr. A and Mr. B decide to meet at a certain restaurant about 8:00 pm. Mr. A arrives at a time uniformly distributed between 7:45 pm and 8:15 pm and Mr. B independently arrives at a time uniformly distributed between 7:30 pm and 8:30 pm.

(i) Find the probability that the first to arrive has to wait more than 10 minutes.

(ii) What is the probability that Mr. A arrives first?

[40 marks]

5. (a) Suppose that $X_1, X_2, ..., X_n$ are observations of a random sample of size n from the normal distribution with mean μ and variance σ^2. Define

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}.$$

(i) Determine the p.d.f. of random variable Y where $Y = S^2$.

(ii) If $n = 25, \mu = 3$ and $\sigma^2 = 16$, find $P(1 < \bar{X} < 4; S^2 < 12)$.

[50 marks]

...7/-

4. (a) (i) *Fungsi penjana momen (f.p.m.) bagi pembolehubah rawak X diberi sebagai*

$$M_X(t) = \frac{1}{3} + \frac{1}{4}e^t + \frac{1}{6}e^{3t} + \frac{1}{4}e^{5t}.$$

Cari $P(0 < X \leq 4)$.

(ii) *F.p.m. bagi pembolehubah rawak X diberi sebagai*

$$M_X(t) = e^{3(e^t - 1)},$$

dan f.p.m. bagi pembolehubah rawak Y diberi sebagai

$$M_Y(t) = \left(0.3e^t + 0.7\right)^4.$$

Cari $E(2X + 3Y + 1)$.

[30 markah]

(b) *Sebuah bekas mengandungi 3 biji guli putih dan 2 biji guli biru. Guli-guli ini dipilih satu demi satu sehingga kedua-dua guli biru diperolehi. Katakan N adalah bilangan percubaan diperlukan sehingga guli biru pertama diperolehi dan M adalah bilangan percubaan tambahan sehingga guli biru yang kedua diperolehi. Dapatkan f.k.k. tercantum bagi N dan M.*

[30 markah]

(c) *En. A dan En. B bercadang untuk bertemu di sebuah restoran pada pukul 8:00 malam. En. A tiba pada masa yang bertaburan seragam antara 7:45 malam dan 8:15 malam manakala En. B secara tak bersandar tiba pada masa yang bertaburan seragam antara 7:30 malam dan 8:30 malam.*

(i) *Dapatkan kebarangkalian bahawa orang pertama yang tiba terpaksa menunggu lebih daripada 10 minit.*

(ii) *Apakah kebarangkalian bahawa En. A tiba dahulu?*

[40 markah]

5. (a) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *adalah cerapan-cerapan daripada sampel rawak bersaiz n daripada taburan normal dengan min* μ *dan varians* σ^2. *Takrifkan*

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad dan \quad S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}.$$

(i) *Tentukan f.k.k. bagi pembolehubah rawak Y yang mana* $Y = S^2$.

(ii) *Jika* $n = 25, \mu = 3$ *dan* $\sigma^2 = 16$, *cari* $P\left(1 < \bar{X} < 4; S^2 < 12\right)$.

[50 markah]

...8/-

(b) Suppose that $X_1, X_2, ..., X_n$ are n mutually independent normal variables with means $\mu_1, \mu_2, ..., \mu_n$ and variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$, respectively. Show that the linear function

$$Y = \sum_{i=1}^{n} a_i X_i$$

where $a_1, a_2, ..., a_n$ are constants, has the normal distribution,

$$N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Hence, compute

$$P(X_1 < X_2)$$

where X_1 and X_2 are independent random variables with distributions $N(2, 4)$ and $N(3, 16)$, respectively.

[50 marks]

(b) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah n pembolehubah-pembolehubah normal yang saling tak bersandar dengan min-min* $\mu_1, \mu_2, ..., \mu_n$ *dan varians-varians* $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$ *, masing-masing. Tunjukkan bahawa fungsi linear*

$$Y = \sum_{i=1}^{n} a_i X_i$$

yang mana $a_1, a_2, ..., a_n$ *adalah pemalar-pemalar, mempunyai taburan normal,*

$$N\left(\sum_{i=1}^{n} a_i \mu_i,\ \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Dengan yang demikian, hitung

$$P(X_1 < X_2)$$

yang mana X_1 *dan* X_2 *ialah pembolehubah-pembolehubah rawak dengan taburan masing-masing N(2, 4) dan N(3, 16).*

[50 markah]

...10/-

APPENDIX

	Probability Density Function
Bernoulli	$p^x(1-p)^{1-x}$, $x = 0,1$, $0 < p < 1$
Binomial	$\binom{n}{x} p^x(1-p)^{n-x}$, $x = 0,1,\ldots,n$, $0 < p < 1$
Hypergeometric	$\dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}$, $x = 0,1,\ldots$, $r \le n$ or $x = 1,2,\ldots$, $n_1 \le r$
Geometric	$(1-p)^{x-1}p$, $x = 1,2,\ldots$
Negative Binomial	$\binom{x-1}{r-1} p^r(1-p)^{x-r}$, $x = r, r+1, \ldots$
Poisson	$\dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,2,\ldots$, $\lambda > 0$
Uniform	$\dfrac{1}{\beta-\alpha}$, $\alpha < x < \beta$
Normal	$\dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$, $-\infty < x < \infty$
Exponential	$\lambda e^{-\lambda x}$, $x \ge 0$
Gamma	$\dfrac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$, $\alpha > 0$
Chi-square	$\dfrac{1}{\Gamma(r/2)\,2^{r/2}} x^{r/2-1} e^{-x/2}$, $x \ge 0$
Beta	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, $\alpha > 0$, $\beta > 0$

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