UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MAT 517 – Computational Linear Algebra and Function Approximation
[Aljabar Linear Pengkomputeran dan Penghampiran Fungsi]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **ELEVEN** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & -0.99 \\ -1 & 1 \end{pmatrix}$.

(a) Use Gaussian elimination to factor $\mathbf{A}$ into the product $\mathbf{LU}$ where $\mathbf{L}$ is a lower triangular matrix whose diagonal entries are 1, and, $\mathbf{U}$ is an upper triangular matrix.

(b) Use the $\mathbf{LU}$ factorization in part (a) to solve the linear equation $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Compare your result with the exact solution to the problem. Discuss on the accuracy of your result and relate with the condition number of $\mathbf{A}$.

(c) Use Householder transformation to produce $\mathbf{QR}$ factorization of $\mathbf{A}$. Use this factorization to solve the same system as in part (b).

(d) You should find that your solution in part (c) is more accurate than the result you have from part (b). Explain this situation by referring to the advantages of Householder transformation compared to Gaussian elimination.

[100 marks]

1. *Diberi matriks* $\mathbf{A} = \begin{pmatrix} 1 & -0.99 \\ -1 & 1 \end{pmatrix}$.

(a) *Guna penghapusan Gauss untuk memfaktorkan* $\mathbf{A}$ *kepada hasil darab* $\mathbf{LU}$ *dengan* $\mathbf{L}$, *matriks segi tiga bawah yang mempunyai pemasukan pepenjuru* 1 *dan* $\mathbf{U}$, *matriks segi tiga atas.*

(b) *Guna faktor* $\mathbf{LU}$ *daripada bahagian (a) untuk menyelesaikan persamaan linear* $\mathbf{Ax} = \mathbf{b}$, *dengan* $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *Bandingkan keputusan anda dengan penyelesaian sebenar masalah tersebut. Bincangkan ketepatan keputusan anda dan hubung kaitkan dengan nombor syarat* $\mathbf{A}$.

(c) *Guna transformasi Householder untuk menghasilkan pemfaktoran* $\mathbf{QR}$ *bagi* $\mathbf{A}$. *Guna pemfaktoran ini untuk menyelesaikan sistem yang sama seperti yang terdapat di bahagian (b).*

(d) *Anda sepatutnya mendapati bahawa penyelesaian di bahagian (c) adalah lebih tepat berbanding keputusan yang anda peroleh di bahagian (b). Terangkan keadaan ini dengan merujuk kepada kelebihan-kelebihan transformasi Householder berbanding dengan penghapusan Gauss.*

[100 markah]

2. Consider finding an orthogonal matrix $\mathbf{H}$ such that the similarity transformation of $\mathbf{A}$ is of the form

$$\mathbf{HAH}^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}_1 & \\ 0 & & & \end{pmatrix},$$

where λ_1 is an eigenvalue of $\mathbf{A}$, and $\mathbf{A}_1$ is the principle submatrix of $\mathbf{HAH}^{-1}$.

(a) Show that $\mathbf{Hx}_1 = \mathbf{e}_1$, where $\mathbf{x}_1$ is an eigenvector of $\mathbf{A}$ associated with λ_1, and, $\mathbf{e}_1$ is the unit Euclidean vector with 1 in the first entry and zero elsewhere. (Hint: Given any matrix $\mathbf{B}$, $\mathbf{Be}_1$ gives the first column of $\mathbf{B}$).

(b) Given the matrix $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

(i) Verify that $\lambda_1 = 2$ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}_1 = (2,1)^T$ is the associated eigenvector.

(ii) Use results from part (a) to find a Householder transformation $\mathbf{H}$ such that $\mathbf{HAH}$ is of the form $\begin{pmatrix} 2 & \times \\ 0 & \times \end{pmatrix}$.

(iii) Compute $\mathbf{HAH}$ and find the other eigenvalue of $\mathbf{A}$.

[100 marks]

2. *Pertimbangkan pencarian matriks berortogon* $\mathbf{H}$ *sebegitu rupa sehinggakan transformasi keserupaan bagi* $\mathbf{A}$ *adalah berbentuk*

$$\mathbf{HAH}^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}_1 & \\ 0 & & & \end{pmatrix},$$

dengan λ_1, *nilai eigen* $\mathbf{A}$ *dan* $\mathbf{A}_1$, *submatriks utama bagi* $\mathbf{HAH}^{-1}$.

(a) *Tunjukkan bahawa* $\mathbf{Hx}_1 = \mathbf{e}_1$, *dengan* $\mathbf{x}_1$ *ialah vektor eigen* $\mathbf{A}$ *yang bersekutu dengan* λ_1, *dan,* $\mathbf{e}_1$ *ialah vektor Euklidan unit dengan 1 berada pada pemasukan pertama dan sifar pada pemasukan lain. (Petunjuk: Diberi sebarang matriks* $\mathbf{B}$, $\mathbf{Be}_1$ *memberikan lajur pertama* $\mathbf{B}$*).*

(b) *Diberi matriks* $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

(i) *Tentu sahkan bahawa* $\lambda_1 = 2$ *ialah nilai eigen* $\mathbf{A}$ *dan* $\mathbf{v}_1 = (2,1)^T$ *ialah vektor eigen yang bersekutu dengannya.*

(ii) *Guna keputusan daripada bahagian (a) untuk mencari transformasi Householder* $\mathbf{H}$ *sebegitu rupa sehinggakan* $\mathbf{HAH}$ *adalah berbentuk* $\begin{pmatrix} 2 & \times \\ 0 & \times \end{pmatrix}$.

(iii) *Kira* $\mathbf{HAH}$ *dan cari nilai eigen* $\mathbf{A}$ *yang lain.*

[100 markah]

3. (a) If $f \in C^1[a,b]$ and $x_0, \cdots, x_n \in [a,b]$ are distinct, show that $H_{2n+1}(x)$, the Hermite polynomial of degree at most $2n+1$, is the unique polynomial of least degree agreeing with f and f' at $x_0, \cdots, x_n \in [a,b]$.
[Hint: Assume that $P(x)$ is another such polynomial with $P(x_k) = f(x_k)$ and $P'(x_k) = f'(x_k)$, for $k = 0, 1, \cdots, n$, and the degree of $P(x)$ is at most $2n+1$. Let $D(x) = H_{2n+1}(x) - P(x)$.]

(b) The data from the observations of a car traveling along a straight road are given in the following table, where the time is in seconds, the distance is in metres, and the speed is in metres per second.

Time	0	3	5
Distance	0	69	117
Speed	22	23	24

(i) Use a Hermite polynomial to predict the position of the car and its speed when t = 4 seconds.

(ii) Use a clamped cubic spline,
$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ for $x \in [x_i, x_{i+1}]$
for each $i = 0, 1,$ in order to obtain a corresponding result to part [i]. Do you think this gives you a better result?

[100 marks]

3. (a) *Jika* $f \in C^1[a,b]$ *dan* $x_0, \cdots, x_n \in [a,b]$ *adalah berbeza, tunjukkan bahawa* $H_{2n+1}(x)$, *polinomial Hermite berdarjah paling tinggi* $2n+1$, *adalah polinomial unik darjah terendah yang sekata dengan* f *dan* f' *pada* $x_0, \cdots, x_n \in [a,b]$.

[Petunjuk: Andaikan $P(x)$ *adalah polinomial seperti itu yang lain dengan* $P(x_k) = f(x_k)$ *dan* $P'(x_k) = f'(x_k)$, *untuk* $k = 0, 1, \cdots, n$, *dan darjah bagi* $P(x)$ *adalah paling tinggi* $2n+1$. *Andaikan juga* $D(x) = H_{2n+1}(x) - P(x)$.*]*

(b) *Data daripada pemerhatian sebuah kereta yang sedang bergerak melalui satu jalan yang lurus diberikan dalam jadual berikut, dengan masa adalah dalam saat, jarak adalah dalam meter, dan kelajuan adalah dalam meter per saat.*

Masa	0	3	5
Jarak	0	69	117
Kelajuan	22	23	24

(i) *Guna polinomial Hermite untuk menjangkakan kedudukan dan kelajuan kereta tersebut apabila* $t = 4$ *saat.*

(ii) *Guna splin kubik tersepit,*

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

bagi $x \in [x_i, x_{i+1}]$ *untuk setiap* $i = 0, 1$, *supaya memperoleh keputusan sepadan dengan di bahagian [i]. Adakah anda fikir keputusan ini lebih baik?*

[100 markah]

4. (a) The following data show a relation between blood sugar levels and cholesterol levels (in mg/dL) for 10 different patients.

	Patient	Blood sugar level (x_i)	Cholesterol level (y_i)	x_i^2	x_iy_i
	1	130	170	16900	22100
	2	138	160	19044	22080
	3	142	173	20164	24566
	4	159	181	25281	28779
	5	165	201	27225	33165
	6	180	215	32400	38700
	7	200	192	40000	38400
	8	210	240	44100	50400
	9	230	260	52900	59800
	10	250	290	62500	72500
$\sum_i$		1804	2082	340514	390490

(i) State the normal equations which are obtained when fitting the best least squares line, $y = a_0x + a_1$ to the above data.

(ii) Thus, find the linear least squares polynomial fitting these data.

(iii) A cholesterol level of below 200 mg/dL is desirable. Based on your result in part [ii], what is the corresponding desirable blood sugar level for a person. Compare your result to the normal blood sugar level, which is below 140 mg/dL.

(b) For $x \in [-1,1]$, the monic Chebyshev polynomials $\{\tilde{T}_n(x)\}$ are derived from the Chebyshev polynomial $T_n(x) = \cos[n \arccos x]$, for each $n \geq 0$.

(i) Write down the set of *monic* Chebyshev polynomials $\{\tilde{T}_2, \tilde{T}_3, \tilde{T}_4, \tilde{T}_5, \tilde{T}_6\}$.

(ii) Show that

$$\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x)$$

and $\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4}\tilde{T}_{n-1}(x)$, for $n \geq 2$.

4. (a) *Data berikut menunjukkan hubungan di antara aras gula darah dan aras kolesterol (dalam mg/dL) bagi 10 pesakit berlainan.*

	Pesakit	*Aras gula darah* (x_i)	*Aras kolesterol* (y_i)	x_i^2	$x_i y_i$
	1	130	170	16900	22100
	2	138	160	19044	22080
	3	142	173	20164	24566
	4	159	181	25281	28779
	5	165	201	27225	33165
	6	180	215	32400	38700
	7	200	192	40000	38400
	8	210	240	44100	50400
	9	230	260	52900	59800
	10	250	290	62500	72500
$\sum_i$		1804	2082	340514	390490

(i) *Nyatakan persamaan normal yang diperoleh apabila memadankan garis kuasa dua terkecil yang terbaik,* $y = a_0x + a_1$ *kepada data di atas.*

(ii) *Cari polinomial kuasa dua terkecil linear yang memadan data ini.*

(iii) *Aras kolesterol di bawah 200 mg/dL adalah sesuatu yang diinginkan. Berdasarkan keputusan kamu di bahagian [ii], apakah aras gula darah sepadan yang diinginkan bagi seseorang. Bandingkan keputusan yang kamu peroleh dengan aras gula darah normal, iaitu di bawah 140 mg/dL.*

(b) *Bagi* $x \in [-1,1]$, *polinomial Chebyshev monik* $\{\tilde{T}_n(x)\}$ *diterbitkan daripada polinomial Chebyshev* $T_n(x) = \cos[n \arccos x]$, *untuk setiap* $n \geq 0$.

(i) *Tuliskan set polinomial Chebyshev monik* $\{\tilde{T}_2, \tilde{T}_3, \tilde{T}_4, \tilde{T}_5, \tilde{T}_6\}$.

(ii) *Tunjukkan bahawa*

$$\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x)$$

$$\text{dan } \tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4}\tilde{T}_{n-1}(x), \text{ bagi } n \geq 2.$$

[iii] The function $f(x) = xe^x$ is approximated on the interval $[-1,1]$ by the sixth Maclaurin polynomial $P_6(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$, which has truncation error $|R_6(x)| = \frac{|f^{(7)}(\xi(x))||x^7|}{5040} \le \frac{e}{630} \approx 0.004315$. Use Chebyshev polynomials to obtain a lesser-degree polynomial approximation while keeping the error less than 0.01 on $[-1,1]$.

[100 marks]

[iii] *Pada selang* $[-1,1]$, *fungsi* $f(x) = xe^x$ *dianggarkan oleh polinomial Maclaurin keenam* $P_6(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$, *yang mempunyai ralat pangkasan* $|R_6(x)| = \frac{|f^{(7)}(\xi(x))||x^7|}{5040} \leq \frac{e}{630} \approx 0.004315$. *Guna polinomial Chebyshev untuk memperoleh hampiran polinomial darjah lebih rendah dengan mengekalkan ralat kurang dari* 0.01 *pada selang* $[-1,1]$.

[100 markah]

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