UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini].

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Given the joint pdf $f_{X,Y}(x,y) = 2e^{-(x+y)}, 0 < x < y < \infty$.
Find
(i) $P(Y < 1 \mid X < 1)$
(ii) $P(Y < 1 \mid X = 1)$
(iii) $f_{Y|x}(y)$

[30 marks]

(b) Let X be a positive continuous random variable with distribution function F and density function f. Find the formula for the density function g for the random variable $Y = \dfrac{1}{1+X}$.

[20 marks]

(c) Let X and Y have joint density function $f(x,y) = 2e^{-(x+y)} I_{(0,y)}(x) I_{(x,\infty)}(y)$. Find the joint density function $S = X$ and $T = X + Y$, marginal density function S and marginal density function T.

[30 marks]

(d) Let X be a random variable having probability density function
$f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}$, $-\infty < x < \infty$. If $Y = X^2$, find the expected value of Y

[20 marks]

2. (a) If $X_1, X_2, \ldots, X_n$ are independent random variables each with common distribution and moment generating function $M_X(t)$. Show that $\bar{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ has moment generating function given by $[M_X(t/n)]^n$.

[20 marks]

(b) A random sample of size $n = 6$ is taken from the pdf $f_Y(y) = 3y^2, 0 \le y \le 1$. Find the probability of the fifth order statistics $P(Y_5' > 0.75)$.

[20 marks]

(c) Suppose a random sample of size n is drawn from the probability model $p_X(k;\theta) = \dfrac{\theta^{2k} e^{-\theta^2}}{k!}$, $k = 0,1,2,\ldots$. Find a formula for the maximum likelihood estimator, $\hat{\theta}$.

[30 marks]

1. (a) *Diberi fungsi ketumpatan kebarangkalian tercantum*

$$f_{X,Y}(x,y) = 2e^{-(x+y)}, 0 < x < y < \infty.$$

Cari

(i) $P(Y < 1 \mid X < 1)$

(ii) $P(Y < 1 \mid X = 1)$

(iii) $f_{Y|x}(y)$

[30 markah]

(b) *Andaikan X suatu pembolehubah rawak selanjar positif dengan fungsi taburan F dan fungsi ketumpatan f. Cari formula bagi fungsi ketumpatan g untuk pembolehubah rawak* $Y = \dfrac{1}{1+X}$.

[20 markah]

(c) *Andaikan X dan Y mempunyai fungsi ketumpatan kebarangkalian tercantum* $f(x,y) = 2e^{-(x+y)} I_{(0,y)}(x)\, I_{(x,\infty)}(y)$.

Cari fungsi ketumpatan tercantum bagi S = X dan T = X + Y, fungsi ketumpatan sut S dan fungsi ketumpatan sut T.

[30 markah]

(d) *Andaikan X suatu pembolehubah rawak dengan fungsi ketumpatan kebarangkalian* $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}, -\infty < x < \infty$. *Jika* $Y = X^2$, *cari nilai jangkaan bagi Y*

[20 markah]

2. (a) *Jika* $X_1, X_2, \ldots, X_n$ *adalah pembolehubah rawak tak bersandar dengan taburan yang sama dan fungsi penjana momen* $M_X(t)$. *Tunjukkan bahawa* $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ *mempunyai fungsi penjana momen* $[M_X(t/n)]^n$.

[20 markah]

(b) *Suatu sampel rawak bersaiz n = 6 diambil dari fungsi ketumpatan kebarangkalian* $f_Y(y) = 3y^2, 0 \le y \le 1$. *Cari kebarangkalian bagi statistik tertib kelima* $P(Y_5' > 0.75)$.

[20 markah]

(c) *Andaikan suatu sampel rawak bersaiz n diambil dari model kebarangkalian* $p_X(k;\theta) = \dfrac{\theta^{2k} e^{-\theta^2}}{k!}$, $k = 0,1,2,\ldots$. *Cari formula bagi penganggar kebolehjadian maksimum,* $\hat{\theta}$.

[30 markah]

(d) Let $X_1, X_2, \ldots, X_n$ denote the outcomes of a series of n independent trials, where $X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases}$ for $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2, + \ldots + X_n$.

(i) Show that $\hat{p}_1 = X_1$ and $\hat{p}_2 = \frac{X}{n}$ are unbiased estimators for p.

(ii) Determine which is a better estimator by comparing the variances of $\hat{p}_1$ and $\hat{p}_2$.

[30 marks]

3. (a) Suppose a random sample of size n is taken from a normal distribution with mean μ and variance σ^2, where σ^2 is known. Compare the Cramer-Rao lower bound for $f_Y(y;\mu)$ with the variance of $\hat{\mu} = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Is $\overline{Y}$ an efficient estimator for μ?

[30 marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the pdf $f_Y(y;\theta) = \frac{1}{(r-1)!\theta^r} y^{r-1} e^{-y/\theta}$, $0 \le y$ for positive parameter θ and r a known positive integer. Find a sufficient statistic for θ.

[20 marks]

(c) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from the exponential pdf, $f_Y(y;\lambda) = \lambda e^{-\lambda y}$, $y > 0$. Show that $\hat{\lambda}_n = Y_1$ is not consistent for λ.

[20 marks]

(d) Assume that $\overline{X}$ denotes the sample mean for a random sample of size n from a N(μ, 9) distribution. Find the value of n so that $(\overline{X} - 1, \overline{X} + 1)$ is a 95% confidence interval for μ.

[30 marks]

(d) *Andaikan* $X_1, X_2, \ldots, X_n$ *melambangkan keputusan suatu siri n percubaan yang tak bersandar, di mana*

$$X_i = \begin{cases} 1, & \text{dengan kebarangkalian } p \\ 0, & \text{dengan kebarangkalian } 1-p \end{cases} \quad \text{bagi } i = 1, 2, \ldots, n.$$

Katakanlah $X = X_1 + X_2, + \ldots + X_n$.

(i) *Tunjukkan bahawa* $\hat{p}_1 = X_1$ *and* $\hat{p}_2 = \dfrac{X}{n}$ *adalah penganggar saksama bagi p.*

(ii) *Tentukan yang mana satu adalah penganggar yang lebih baik dengan membanding varians bagi* $\hat{p}_1$ *and* $\hat{p}_2$.

[30 markah]

3. (a) *Andaikan suatu sampel rawak bersaiz n di ambil dari suatu taburan normal dengan min* μ *dan varians* σ^2, *di mana* σ^2 *diketahui. Bandingkan batas bawah Cramer-Rao bagi* $f_Y(y;\mu)$ *dengan varians untuk* $\hat{\mu} = \overline{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$. *Adakah* $\overline{Y}$ *suatu penganggar cekap bagi* μ?

[30 markah]

(b) *Katakanlah* $Y_1, Y_2, \ldots, Y_n$ *suatu sampel rawak bersaiz n dari fungsi ketumpatan kebarangkalian* $f_Y(y;\theta) = \dfrac{1}{(r-1)!\theta^r} y^{r-1} e^{-y/\theta}$, $0 \le y$ *bagi parameter positif* θ *and r suatu integer positif yang diketahui. Cari suatu statistik cukup bagi* θ.

[20 markah]

(c) *Andaikan* $Y_1, Y_2, \ldots, Y_n$ *suatu sampel rawak dari fungsi ketumpatan kebarangkalian eksponen,* $f_Y(y;\lambda) = \lambda e^{-\lambda y}$, $y > 0$. *Tunjukkan bahawa* $\hat{\lambda}_n = Y_1$ *tidak konsisten bagi* λ.

[20 markah]

(d) *Andaikan bahawa* $\overline{X}$ *melambangkan min sampel bagi sampel rawak bersaiz n dari taburan* $N(\mu, 9)$. *Cari nilai n supaya* $(\overline{X} - 1, \overline{X} + 1)$ *ialah selang keyakinan 95% bagi* μ.

[30 markah]

4. (a) Assume that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution having pdf $f(x;\theta) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$; $-\infty < \theta < \infty$.
For testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$, the following test is used:
Reject H_0 if and only if $Y_1 = \min(X_1, X_2, \ldots, X_n) > c$.
Find the power function of this test. If the size of this test is α, find c in terms of α.

[40 marks]

(b) Let $X_1, X_2, \ldots, X_{10}$ be a random sample of size 10 from a normal distribution $N(0, \sigma^2)$. Find a best critical region of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 2$.

[30 marks]

(c) Let X have the pdf $f_X(x;\theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1,$ zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta < \frac{1}{2}$ by taking a random sample $X_1, X_2, \ldots, X_5$ of size n = 5 and rejecting H_0 if $Y = \sum_{1}^{n} X_i$ is observed to be less than or equal to a constant c.
(i) Show that this is a uniformly most powerful test.
(ii) Find the significance level when c = 1.
(iii)Find the significance level when c = 0.

[30 marks]

4. (a) *Andaikan bahawa* $X_1, X_2, \ldots, X_n$ *adalah sampel rawak dari suatu taburan yang mempunyai fungsi ketumpatan kebarangkalian* $f(x;\theta) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$; $-\infty < \theta < \infty$.
Bagi menguji $H_0 : \theta = \theta_0$ *lawan.* $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$, *ujian berikut digunakan:*
Tolak H_0 *jika dan hanya jika* $Y_1 = \min(X_1, X_2, \ldots, X_n) > c$.
Cari fungsi kuasa bagi ujian ini. Jika saiz bagi ujian adalah α, *cari c dalam sebutan* α.

[40 markah]

(b) *Katakanlah* $X_1, X_2, \ldots, X_{10}$ *adalah sampel rawak bersaiz 10 dari taburan normal* $N(0,\sigma^2)$. *Cari rantau genting terbaik yang bersaiz* $\alpha = 0.05$ *bagi menguji* $H_0 : \sigma^2 = 1$ *lawan* $H_1 : \sigma^2 = 2$.

[30 markah]

(c) *Katakanlah* X *mempunyai fungsi ketumpatan kebarangkalian* . $f_X(x;\theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$, *sifar di tempat lain. Kita menguji* $H_0 : \theta = \frac{1}{2}$ *lawan* $H_1 : \theta < \frac{1}{2}$ *dengan mengambil sampel rawak* $X_1, X_2, \ldots, X_5$ *bersaiz* $n = 5$ *dan menolak* H_0 *jika* $Y = \sum_{1}^{n} X_i$ *diperhatikan sebagai kurang atau sama dengan suatu pemalar c.*
(i) *Tunjukkan bahawa ujian ini adalah ujian paling berkuasa secara seragam.*
(ii) *Cari aras keertian apabila c = 1.*
(iii) *Cari aras keertian apabila c = 0.*

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\ldots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\ldots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\ldots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\ldots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

- 000 O 000 -