UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

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MAT 263 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) A health centre offers a 1-year diet program which consists of the basic and advanced modules, each need 6 months to complete. A total of 40 percent of the women and 25 percent of the men managed to reduce their weight at least 5 kg after completing the basic module. These people are given voucher when they continue with the advanced module. If 70 percent of the people attending the 1-year diet program were female,

   (i) what percentage of the people attending the basic module are given voucher when they continue with the advanced module?

   (ii) what percentage of those given the voucher were men?

   [30 marks]

   (b) Suppose that $P(X=0)=1-P(X=1)$ and $E(X)=4\ \mathrm{Var}(X)$. Find $P(X=0)$.

   [20 marks]

   (c) A random variable X has a Poisson distribution with parameter λ. A random variable, Y also has a Poisson distribution and the probability that $Y = i$ is given by $P(Y=i)=P(X=i \mid X>0)$. Find $E(Y)$.

   [30 marks]

   (d) State the Chebyshev's Inequality. If X is a random variable with $E(X)=3$ and $E(X^2)=10$, find the lower bound for $P(1<X<5)$.

   [20 marks]

2. (a) The resistance, X of an electrical component has a probability density function (p.d.f.) defined as

$$f(x)=\begin{cases} Cx(100-x^2) & , \ 20 \le x \le 22 \\ 0 & , \ \text{elsewhere.} \end{cases}$$

   (i) Find the constant C.

   (ii) Determine the cumulative distribution function (c.d.f.), $F(x)$.

   (iii) Find the median of the resistance.

   (iv) Suppose that three of the components are used in an electrical circuit so that the total resistance is the sum of the individual resistances. It is assumed that the resistances of the three components are independent of each other. Find the expected value and the standard deviation of the total resistance.

   [40 marks]

1. (a) *Sebuah pusat kesihatan menawarkan program diet 1-tahun yang terbahagi kepada dua modul, asas dan lanjutan, setiap satunya mengambil masa enam bulan untuk tamat. Sejumlah 40 peratus wanita dan 25 peratus lelaki berjaya menurunkan berat badan sekurang-kurangnya 5 kg selepas menamatkan modul asas. Mereka yang berjaya ini diberi baucer apabila mereka menyambung ke modul lanjutan. Jika 70 peratus yang menyertai program diet 1-tahun ini adalah wanita,*

   (i) *apakah peratusan mereka yang menyertai modul asas diberi baucer apabila menyambung ke modul lanjutan?*

   (ii) *apakah peratusan mereka yang diberi baucer ini lelaki?*

   [30 markah]

   (b) *Andaikan* $P(X=0)=1-P(X=1)$ *dan* $E(X) = 4\,Var(X)$. *Dapatkan* $P(X = 0)$.

   [20 markah]

   (c) *Suatu pembolehubah rawak X mempunyai taburan Poisson dengan parameter* λ. *Suatu pembolehubah rawak, Y juga mempunyai taburan Poisson dan kebarangkalain bahawa* $Y = i$ *diberi oleh* $P(Y=i)=P(X=i \mid X>0)$. *Dapatkan* $E(Y)$.

   [30 markah]

   (d) *Nyatakan Ketaksamaan Chebyshev. Jika X ialah suatu pembolehubah rawak dengan* $E(X)=3$ *dan* $E(X^2)=10$, *dapatkan batas bawah bagi* $P(1<X<5)$.

   [20 markah]

2. (a) *Rintangan, X bagi suatu komponen elektrik mempunyai fungsi ketumpatan kebarangkalian (f.k.k.) yang ditakrif sebagai*

$$f(x)=\begin{cases} Cx(100-x^2), & 20 \le x \le 22 \\ 0, & \text{di tempat lain.} \end{cases}$$

   (i) *Dapatkan nilai pemalar C.*

   (ii) *Tentukan fungsi taburan longgokan (f.t.l), F(x).*

   (iii) *Dapatkan median bagi rintangan.*

   (iv) *Andaikan bahawa tiga daripada komponen-komponen ini digunakan dalam suatu litar elektrik dan jumlah rintangan adalah hasil tambah rintangan-rintangan individu. Diandaikan bahawa rintangan ketiga-tiga komponen adalah tidak bersandar antara satu sama lain. Dapatkan nilai jangkaan dan sisihan piawai bagi jumlah rintangan.*

   [40 markah]

(b) The p.d.f. of the random variable X is given by

$$f(x)=\begin{cases} e^{-(x-4)} & . \quad x \geq 4 \\ 0 & , \text{ elsewhere.} \end{cases}$$

Find the p.d.f. of the random variable $Y = X^2$.

[30 marks]

(c) Suppose that the random variables X and Y are independent. The moment generating function (m.g.f.) of X is $M_X(t)=\frac{1}{4}e^{2t}+\frac{1}{2}e^{3t}+\frac{1}{4}e^{5t}$ and the m.g.f. of Y is $M_Y(t)=e^{3(e^t-1)}$.

(i) Find the p.d.f. and the expected value of X.

(ii) Determine the m.g.f. of a random variable $Z = X + Y$.

[30 marks]

3. (a) Let X and Y have the joint p.d.f. defined as

$y \backslash x$	1	2	3
1	0.10	0.15	0.05
2	0.25	0.20	0.25

Find

(i) marginal p.d.f. of X,

(ii) conditional p.d.f. of Y given $X = 2$,

(iii) $E(Y \mid X = 2)$,

(iv) $P(X - Y < 1)$.

[30 marks]

(b) If

$$f(x,y)=\begin{cases} k\left(x^2+\dfrac{xy}{2}\right) & , \ 0<x<1, 0<y<2 \\ 0 & , \quad \text{elsewhere,} \end{cases}$$

(i) find the constant k so that $f(x,y)$ is a joint p.d.f. of X and Y.

(ii) Find $P\left(Y>1 \,\middle|\, X>\frac{1}{2}\right)$.

(iii) Find $P(X+Y<1)$.

[40 marks]

(b) *F. k.k. bagi pembolehubah rawak X diberi sebagai*

$$f(x) = \begin{cases} e^{-(x-4)} & . \quad x \geq 4 \\ 0 & , \text{ di tempat lain.} \end{cases}$$

Dapatkan f.k.k. bagi pembolehubah rawak $Y = X^2$.

[30 markah]

(c) *Andaikan bahawa pembolehubah-pembolehubah rawak X dan Y adalah tak bersandar. Fungsi penjana momen (f.p.m.) bagi X ialah* $M_X(t) = \frac{1}{4}e^{2t} + \frac{1}{2}e^{3t} + \frac{1}{4}e^{5t}$ *dan f.p.m. bagi Y ialah* $M_Y(t) = e^{3(e^t-1)}$.

(i) *Dapatkan f.k.k. dan nilai jangkaan bagi X.*

(ii) *Tentukan f.p.m. bagi suatu pembolehubah rawak Z = X + Y.*

[30 markah]

3. (a) *Biar X dan Y mempunyai f.k.k. tercantum ditakrif sebagai*

y \ x	1	2	3
1	0.10	0.15	0.05
2	0.25	0.20	0.25

Dapatkan

(i) *f.k.k. sut bagi X,*

(ii) *f.k.k. bersyarat bagi Y diberi X = 2,*

(iii) $E(Y \mid X = 2)$,

(iv) $P(X - Y < 1)$.

[30 markah]

(b) *Jika*

$$f(x,y) = \begin{cases} k\left(x^2 + \frac{xy}{2}\right), & 0 < x < 1, 0 < y < 2 \\ 0 & , \text{ di tempat lain,} \end{cases}$$

(i) *dapatkan nilai pemalar k supaya* $f(x,y)$ *ialah suatu f.k.k. tercantum bagi X and Y.*

(ii) *Dapatkan* $P\left(Y > 1 \middle| X > \frac{1}{2}\right)$.

(iii) *Dapatkan* $P(X + Y < 1)$.

[40 markah]

(c) Let X, Y be two independent standard normal random variables. Show that a random variable $U = X/Y$ has a Cauchy distribution with p.d.f. defined as

$$f(u) = \frac{1}{\pi\left(1+u^2\right)}, \quad -\infty < u < \infty.$$

[Hint: Use an auxiliary variable, V and determine the marginal p.d.f. of U from the joint p.d.f. of U and V, $f(u,v)$.]

[30 marks]

4. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with m.g.f., $M_X(t)$.

Show that

(i) the m.g.f. of $Y = \sum_{i=1}^{n} X_i$ is $[M_X(t)]^n$, and

(ii) the m.g.f. of $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is $\left[M_X\left(\frac{t}{n}\right)\right]^n$.

[30 marks]

(b) $X_1, X_2, \ldots, X_{16}$ is a random sample of size $n = 16$ from the normal distribution with mean 12 and variance 9. The statistics $\overline{X}$ and S^2 are defined as

$$\overline{X} = \frac{\sum_{i=1}^{16} X_i}{16} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{16}\left(X_i - \overline{X}\right)^2}{15},$$

respectively.

Find $P\left(9 < \overline{X} < 15,\ S^2 < 8\right)$.

[20 marks]

(c) Let X_1, X_2 be a random sample of size $n = 2$ from the exponential distribution with p.d.f., $f(x)$ defined by

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Define $U = X_1 + X_2$ and $V = \frac{X_1}{X_1 + X_2}$.

(i) Find the joint p.d.f. of U and V, $f(u,v)$.

(ii) Determine the marginal p.d.f. of U.

(iii) Are U and V independent?

[50 marks]

(c) *Biar X, Y sebagai dua pembolehubah rawak normal piawai yang tak bersandar. Tunjukkan bahawa suatu pembolehubah rawak* $U = X/Y$ *mempunyai taburan Cauchy dengan f.k.k. ditakrif sebagai*

$$f(u) = \frac{1}{\pi\left(1+u^2\right)}, \quad -\infty < u < \infty .$$

[Petunjuk: Gunakan pembolehubah bantu, V dan tentukan f.k.k. sut bagi U daripada f.k.k. tercantum bagi U dan V, $f(u,v)$ *.]*

[30 markah]

4. (a) *Biar* $X_1, X_2, ..., X_n$ *sebagai suatu sampel rawak daripada taburan dengan f.p.m.,* $M_X(t)$.
Tunjukkan bahawa

(i) *f.p.m. bagi* $Y = \sum_{i=1}^{n} X_i$ *ialah* $\left[M_X(t)\right]^n$, *dan*

(ii) *f.p.m. bagi* $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ *ialah* $\left[M_X\left(\dfrac{t}{n}\right)\right]^n$.

[30 markah]

(b) $X_1, X_2, ..., X_{16}$ *ialah suatu sampel rawak bersaiz n = 16 daripada taburan normal dengan min 12 and varians 9. Statistik* $\overline{X}$ *dan statistik* S^2 *ditakrifkan masing-masing sebagai*

$$\overline{X} = \frac{\sum_{i=1}^{16} X_i}{16} \quad dan \quad S^2 = \frac{\sum_{i=1}^{16}\left(X_i - \overline{X}\right)^2}{15}.$$

Dapatkan $P\left(9 < \overline{X} < 15 ,\ S^2 < 8\right)$.

[20 markah]

(c) *Biar* X_1, X_2 *sebagai suatu sampel rawak bersaiz n = 2 daripada taburan eksponen dengan f.k.k.,* $f(x)$ *ditakrif sebagai*

$$f(x) = \begin{cases} e^{-x} , & x > 0 \\ 0 , & \text{di tempat lain.} \end{cases}$$

Takrifkan $U = X_1 + X_2$ *dan* $V = \dfrac{X_1}{X_1 + X_2}$.

(i) *Dapatkan f.k.k. tercantum bagi U dan V,* $f(u,v)$.

(ii) *Tentukan f.k.k. sut bagi U.*

(iii) *Adakah U dan V tak bersandar?*

[50 markah]

APPENDIX

	Probability Density Function
Bernoulli	$p^x(1-p)^{1-x}$, $x=0,1$, $0<p<1$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,\ldots,n$, $0<p<1$
Hypergeometric	$\dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n_1+n_2}{r}}$, $x=0,1,\ldots$, $r\le n$ or $x=1,2,\ldots$, $n_1\le r$
Geometric	$(1-p)^{x-1}p$, $x=1,2,\ldots$
Negative Binomial	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$, $x=r,r+1,\ldots$
Poisson	$\dfrac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\ldots$, $\lambda>0$
Uniform	$\dfrac{1}{\beta-\alpha}$, $\alpha<x<\beta$
Normal	$\dfrac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$, $-\infty<x<\infty$
Exponential	$\lambda e^{-\lambda x}$, $x\ge 0$
Gamma	$\dfrac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}$, $x\ge 0$, $\lambda>0$, $\alpha>0$
Chi-square	$\dfrac{1}{\Gamma(r/2)\,2^{r/2}}x^{r/2-1}e^{-x/2}$, $x\ge 0$
Beta	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1$, $\alpha>0$, $\beta>0$

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