UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MAT 222 – Differential Equations II
[Persamaan Pembezaan II]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

*[**Arahan** : Jawab **semua empat** [4] soalan.]*

1. (a) Consider the vectors $\mathbf{X}_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

(i) Compute the Wronskian of $\mathbf{X}_1$ and $\mathbf{X}_2$.

(ii) In what intervals of t are $\mathbf{X}_1$ and $\mathbf{X}_2$ linearly independent?

(iii) Find the system of homogeneous differential equation satisfied by $\mathbf{X}_1$ and $\mathbf{X}_2$. How do the coefficients of the system explain your results in (ii)?

[40 marks]

(b) Consider the linear system of homogeneous equation

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix}\mathbf{X}. \qquad (1)$$

(i) Verify that vectors $\mathbf{X}_1 = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} 2te^{-2t} \\ (4t-1)e^{-2t} \end{pmatrix}$ are linearly independent solutions of (1). Thus, write the general solution of the system and describe the behavior of the solutions as $t \to \infty$.

(ii) Use variation of parameters to solve the nonhomogeneous linear system of equation

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\frac{e^{-2t}}{t}.$$

[60 marks]

1. (a) *Pertimbangkan vektor* $\mathbf{X}_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ dan $\mathbf{X}_2 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

(i) *Nilaikan Wronskian bagi* $\mathbf{X}_1$ *dan* $\mathbf{X}_2$.

(ii) *Dalam selang* t *yang manakah* $\mathbf{X}_1$ *dan* $\mathbf{X}_2$ *tidak bersandar linear?*

(iii) *Cari sistem persamaan pembezaan linear homogen yang dipenuhi oleh* $\mathbf{X}_1$ *dan* $\mathbf{X}_2$. *Bagaimanakah pekali-pekali sistem tersebut menerangkan tentang keputusan anda dalam (ii)?*

[40 markah]

(b) *Pertimbangkan sistem persamaan linear homogen*

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \mathbf{X}. \qquad (1)$$

(i) *Tentusahkan bahawa vektor* $\mathbf{X}_1 = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$ *dan* $\mathbf{X}_2 = \begin{pmatrix} 2te^{-2t} \\ (4t-1)e^{-2t} \end{pmatrix}$ *penyelesaian-penyelesaian bagi (1) yang tidak bersandar linear. Maka, tulis penyelesaian am sistem tersebut dan huraikan tabiat penyelesaian apabila* $t \to \infty$.

(ii) *Guna ubahan parameter untuk menyelesaikan sistem persamaan linear tak homogen*

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{e^{-2t}}{t}.$$

[60 markah]

2. (a) Consider the linear plane autonomous system

$$\begin{aligned}\frac{dx}{dt} &= \mu x + y \\ \frac{dy}{dt} &= -x + y\end{aligned} \qquad (2)$$

where μ is a real constant and $\mu \neq 1$.

(i) Show that $(0,0)$ is always an unstable critical point of the system (2).

(ii) When is $(0,0)$ an unstable saddle point?

(iii) When is $(0,0)$ an unstable spiral point?

[40 marks]

(b) Given the plane autonomous system

$$\begin{aligned}\frac{dx}{dt} &= 2x - y^2 \\ \frac{dy}{dt} &= -y + xy\end{aligned} \qquad (3)$$

(i) Find all critical points of the system in (3).

(ii) By referring to the values of trace and determinant of the Jacobian matrix of the system, classify the critical points as a stable node, a stable spiral point, an unstable spiral point, an unstable node, or, a saddle point.

[60 marks]

2. (a) *Pertimbangkan sistem satah berautonomi*

$$\frac{dx}{dt} = \mu x + y$$
$$\frac{dy}{dt} = -x + y \qquad (2)$$

di mana μ *ialah pemalar nyata dan* $\mu \neq 1$.

(i) *Tunjukkan bahawa* $(0,0)$ *sentiasa satu titik kritikal sistem (2) yang tidak stabil.*

(ii) *Bilakah* $(0,0)$ *satu titik pelana tidak stabil?*

(iii) *Bilakah* $(0,0)$ *satu titik pusaran tidak stabil?*

[40 markah]

(b) *Diberi sistem satah berautonomi*

$$\frac{dx}{dt} = 2x - y^2$$
$$\frac{dy}{dt} = -y + xy \qquad (3)$$

(i) *Cari semua titik kritikal bagi sistem (3).*

(ii) *Dengan merujuk kepada nilai surihan dan penentu matriks Jacobi sistem tersebut, klasifikasikan titik-titik kritikalnya sebagai nod stabil, pusaran stabil, pusaran tak stabil, nod tak stabil atau titik pelana.*

[60 markah]

3. (a) Find the Fourier series representation for $f(x)=\sqrt{1-\cos x}$ in $-\pi<x<\pi$.

[30 marks]

(b) Find the half range Fourier sine series for sinh (ax) in $0<x<\pi$.

[20 marks]

(c) Consider $y''+\lambda y=0$ subject to the periodic boundary conditions $y(-L)=y(L)$, $y'(-L)=y'(L)$.Show that the eigen functions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots\ldots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \ldots.\right\}$$

Show that the set is orthogonal on [-L,L].

[50 marks]

3. (a) *Cari perwakilan siri Fourier untuk* $f(x)=\sqrt{1-\cos x}$ *in* $-\pi < x < \pi$.

[30 markah]

(b) *Cari siri Fourier separuh julat untuk* $\sinh(ax)$ *dalam* $0 < x < \pi$.

[20 markah]

(c) *Pertimbangkan* $y'' + \lambda y = 0$ *tertakluk kepada syarat sempadan berkala* $y(-L) = y(L)$, $y'(-L) = y'(L)$. *Tunjukkan bahawa fungsi-fungsi eigennya adalah*

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots\ldots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \ldots.\right\}.$$

Tunjukkan bahawa set ini berortogon pada [-L,L].

[50 markah]

4. (a) Use separation of variables to find the product solutions of

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

[20 marks]

(b) Classify the partial differential equation

$$x^2 u_{xx} + 2xyu_{xy} + (1+y^2)u_{yy} - 2u_x = 0$$

[10 marks]

(c) Consider the wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; 0 < x < L, t > 0$$

subject to boundary condition $u(0,t) = 0, u(L,t) = 0,\ t > 0$ and initial conditions

$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t} = g(x),$$

at $t = 0, 0 < x < L$.

(i) Describe how you can obtain a solution to the boundary value problem above using separation of variables.

(ii) Solve the wave equation when

$$f(x) = \begin{cases} \dfrac{2hx}{L}, 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), \dfrac{L}{2} \le x < L \end{cases}; g(x) = 0.$$

[70 marks]

4. (a) *Gunakan pemisahan pembolehubah untuk mencari penyelesaian hasildarab bagi*

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

[20 markah]

(b) *Klasifikasikan persamaan pembezaan separa*

$$x^2 u_{xx} + 2xyu_{xy} + (1+y^2)u_{yy} - 2u_x = 0.$$

[10 markah]

(c) *Pertimbangkan persamaan gelombang*

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; 0 < x < L, t > 0$$

tertakluk kepada syarat sempadan $u(0,t) = 0, u(L,t) = 0,\ t > 0$, *dan, syarat-syarat awal*

$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t} = g(x),$$

pada $t = 0,\ 0 < x < L$.

(i) *Huraikan bagaimana anda boleh mendapatkan satu penyelesaian kepada masalah nilai sempadan di atas menggunakan pemisahan pembolehubah.*

(ii) *Selesaikan persamaan gelombang di atas bila*

$$f(x) = \begin{cases} \dfrac{2hx}{L}, 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), \dfrac{L}{2} \le x < L \end{cases}; g(x) = 0.$$

[70 markah]

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