UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

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MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini].

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Given

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}, B=\begin{bmatrix}1&-1&1\\1&0&1\\0&0&1\end{bmatrix}, C=\begin{bmatrix}1&2&0\\-3&1&1\\5&-1&0\end{bmatrix}$$

Compute:

$$3A^2-4B^2-6BA+2AB-2BC+AC$$

using properties of matrix operations.

(b) (i) Suppose a square matrix A satisfies $A^3+A+I=\underline{0}$. Show that A is non-singular.

(ii) Define a symmetric matrix. Find an example where A and B are symmetric matrices of the same size but AB is not symmetric.
[Hint: Let A and B be in $M_{2\times 2}$]

(c) By inspection, determine whether the following sets of vectors are linearly dependent or linearly independent. (Justify your answer.)

(i) $\{(-4,0,1,5),(0,0,0,0),(0,4,3,6)\}$

(ii) $\{(4,4),(-1,3,),(2,5),(8,1)\}$

(iii) $\{(-8,12,-4),(2,-3,-1)\}$

(d) Suppose that A is the standard matrix representing a linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ where

$$A=\begin{bmatrix}1&-2\\4&-7\end{bmatrix}$$

Find the inverse of A using the concept of linear transformation.

[100 marks]

2. (a) (i) Let $v_1, v_2, ..., v_n$ be elements of a vector space V. Complete the following statement: "A linear combination of $v_1, v_2, ..., v_n$ is ....."

(ii) Write (x,y,z) in $\mathbb{R}^3$ as a linear combination of $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$.

(b) Given that $L(v,w)$ denotes the linear span of the vectors v and w. Consider the two vectors:

$$v=(4,2,-1) \text{ and } w=(-3,1,2).$$

(i) Either prove that $u=(1,3,1)$ is in $L(v,w)$ or prove that is not.

(ii) By (i), are the three vectors u, v, and w linearly independent or linearly dependent? Explain.

[100 marks]

1. (a) *Diberi*

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}, B=\begin{bmatrix}1&-1&1\\1&0&1\\0&0&1\end{bmatrix}, C=\begin{bmatrix}1&2&0\\-3&1&1\\5&-1&0\end{bmatrix}$$

Kirakan:

$$3A^2-4B^2-6BA+2AB-2BC+AC$$

menggunakan sifat-sifat operasi matriks.

(b) (i) *Andai suatu matriks memenuhi* $A^3+A+I=\underline{0}$. *Tunjukkan bahawa A adalah tak singular.*

(ii) *Takrifkan matriks simetri. Cari suatu contoh matriks simetri A dan B yang sama saiz tetapi AB tak simetri.*
[Petunjuk: Biar A dan B dalam $M_{2\times2}$*]*

(c) *Dengan pemeriksaan, tentukan samada vektor-vektor berikut adalah bersandar linear atau tak bersandar linear. (Sahkan jawapan anda)*

(i) $\{(-4,0,1,5),(0,0,0,0),(0,4,3,6)\}$

(ii) $\{(4,4),(-1,3,),(2,5),(8,1)\}$

(iii) $\{(-8,12,-4),(2,-3,-1)\}$

(d) *Andai A ialah matriks piawai mewakili* $T:\mathbb{R}^2\to\mathbb{R}^2$ *yang mana*

$$A=\begin{bmatrix}1&-2\\4&-7\end{bmatrix}$$

Cari songsang A menggunakan konsep transformasi linear.

[100 markah]

2. (a) (i) *Biar* $v_1,v_2,\ldots,v_n$ *unsur-unsur vektor V. Lengkapkan kenyataan berikut:*
"Suatu gabungan linear $v_1,v_2,\ldots,v_n$ *ialah ....."*

(ii) *Tulis* (x,y,z) *dalam* $\mathbb{R}^3$ *sebagai gabungan linear* $(1,1,1)$, $(1,2,3)$ *dan* $(2,-1,1)$.

(b) *Diberi* $L(v,w)$ *menandakan rentangan linear vektor v dan w. Pertimbangkan dua vektor:*

$$v=(4,2,-1) \text{ dan } w=(-3,1,2).$$

(i) *Sama ada buktikan* $u=(1,3,1)$ dalam $L(v,w)$ *atau buktikan sebaliknya.*

(ii) *Dengan (i), adakah tiga vektor u, v, dan w bersandar linear atau tak bersandar linear? Jelaskan.*

[100 markah]

(c) Given that $v = L(u_1, u_2, u_3, u_4)$ in $\mathbb{R}^3$ where

$$u_1 = (1,0,1), u_2 = (2,1,2,), u_3 = (0,0,1) \text{ and } u_4 = (1,1,1)$$

(i) Find a set $S \subseteq \{u_1, u_2, u_3, u_4\}$ such that S is linearly independent and $L(S) = L(u_1, u_2, u_3, u_4)$.

(ii) Deduce the dimension of $L(u_1, u_2, u_3, u_4)$.

(d) Consider the matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -6 & 3 & 6 \\ 3 & 1 & 7 \end{bmatrix}$.

(i) Define the row space of A.

(ii) Find the basis and dimension of the row space of A.

[100 marks]

3. (a) Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x_1, x_2)T = (5x_2 - 3, 7x_1 + x_2)$ is not a linear transformation.

(b) Let T be a linear transformation with $(e_1 + 2e_2 + e_3)T = (8,1,7)$, $(e_1 + e_3) = (4,3,7)$ and $(e_3)T = (3,1,4)$

(i) Show that $\{e_1 + 2e_2 + e_3, e_1 + e_3, e_3\}$ is linearly independent.

(ii) Show that

$$(x, y, z) = \frac{y}{2}(e_1 + 2e_2 + e_3) + \left(x - \frac{y}{2}\right)(e_1 + e_3) + (z - x)(e_3)$$

Thus, or otherwise, find the definition of T.

(c) Show that each of the following set V is not a vector space by giving a counterexample for one axiom that V does not satisfy (Do not show more than one axiom!)

(i) $V = \{p(x) \in P_2(\mathbb{R})\}$ with the operations

$$(ax + b) \oplus (cx + d) = (a + c) + (b + d)$$

$$k \odot (ax + b) = 2kax + 2kb, k \in \mathbb{R}$$

(ii) $V = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$ with the usual addition and scalar multiplication operations in $\mathbb{R}^2$.

(iii) $V = \{A \in M_{2\times2} \mid A = [a_{ij}] \text{ and } a_{11}\, a_{22} \le 0, a_{ij} \in \mathbb{R}\}$ with the usual addition and scalar multiplication operations in $M_{2\times2}$.

(d) Given $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$(x, y, z, w)T = (x - 2y + z + w, -x + 2y + w, 2x - 4y + z)$$

(i) Use the Gauss Jordan process in finding the kernel of T.

(ii) Determine the basis and dimension of the kernel of T from your result in (i).

(iii) Find the basis and dimension of the image of T.

(iv) From your result in (ii) and (iii), verify the Dimension Theorem.

[100 marks]

(c) *Diberi* $v = L(u_1, u_2, u_3, u_4)$ *dalam* $\mathbb{R}^3$ *yang mana*

$$u_1 = (1,0,1), u_2 = (2,1,2,), u_3 = (0,0,1) \text{ dan } u_4 = (1,1,1)$$

(i) *Cari set* $S \subseteq \{u_1, u_2, u_3, u_4\}$ *sedemikian hingga* *S tak bersandar linear dan* $L(S) = L(u_1, u_2, u_3, u_4)$.

(ii) *Deduksikan dimensi* $L(u_1, u_2, u_3, u_4)$.

(d) *Pertimbangkan matriks* $A = \begin{bmatrix} 4 & -1 & 0 \\ -6 & 3 & 6 \\ 3 & 1 & 7 \end{bmatrix}$.

(i) *Takrifkan ruang baris A.*

(ii) *Cari asas dan dimensi ruang baris A.*

[100 markah]

3. (a) *Tunjukkan bahawa* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *ditakrifkan dengan* $(x_1, x_2)T = (5x_2 - 3, 7x_1 + x_2)$ *bukan suatu transformasi linear.*

(b) *Biar T suatu transformasi linear dengan* $(e_1 + 2e_2 + e_3)T = (8,1,7)$, $(e_1 + e_3) = (4,3,7)$ *dan* $(e_3)T = (3,1,4)$

(i) *Tunjukkan bahawa* $\{e_1 + 2e_2 + e_3, e_1 + e_3, e_3\}$ *adalah tak bersandar linear.*

(ii) *Tunjukkan bahawa*

$$(x, y, z) = \frac{y}{2}(e_1 + 2e_2 + e_3) + \left(x - \frac{y}{2}\right)(e_1 + e_3) + (z - x)(e_3)$$

Dengan itu, atau cara lain, cari takrif bagi T.

(c) *Tunjukkan bahawa setiap daripada set V berikut bukan suatu ruang vektor dengan memberikan suatu contoh lawan untuk satu axiom yang V tidak penuhi. (Jangan tunjuk lebih dari satu axiom!)*

(i) $V = \{p(x) \in P_2(\mathbb{R})\}$ *dengan operasi*

$(ax + b) \oplus (cx + d) = (a + c) + (b + d)$

$k \odot (ax + b) = 2kax + 2kb, k \in \mathbb{R}$

(ii) $V = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$ *dengan operasi penambahan dan pendaraban skalar biasa dalam* $\mathbb{R}^2$.

(iii) $V = \{A \in M_{2\times2} \mid A = [a_{ij}] \text{ and } a_{11}a_{22} \le 0, a_{ij} \in \mathbb{R}\}$ *dengan operasi penambahan dan pendaraban biasa dalam* $M_{2\times2}$.

(d) *Diberi* $T : \mathbb{R}^4 \to \mathbb{R}^3$ *tertakrif dengan*

$(x, y, z, w)T = (x - 2y + z + w, -x + 2y + w, 2x - 4y + z)$

(i) *Guna proses Gauss Jordan dalam mencari inti bagi T.*

(ii) *Tentukan asas dan dimensi inti bagi T dari keputusan anda dalam (i).*

(iii) *Cari asas dan dimensi imej dari T.*

(iv) *Dari keputusan anda dalam (ii) dan (iii), tentusahkan Teorem Dimensi.*

[100 markah]

4. (a) Given a subspace $W = L(u_1, u_2 u_3)$ where

$$u_1 = (1,1,0,0),\ u_2 = (2,-1,0,0),\ u_3 = (3,-3,0,-2)$$

Find on orthonormal basis of W using the Gram-Schmidt process.

(b) (i) Let U be a subspace of a vector space V. If $U^{\perp}$ denotes the orthogonal complement of U, show that the only case in which a vector x can be in both W and $W^{\perp}$ is when $x = \underline{0}$.

(ii) Show that if W is a subspace of $\mathbb{R}^n$ then $W^{\perp} = \mathbb{R}^n$ if and only if $W = \{\underline{0}\}$.

(c) Find the best-fit line for the points

$$(-2,1), (-1,3), (0,2), (1,3), (2,1).$$

(d) (i) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations defined by $(x,y)T = (x-2y,-y)$ where $\alpha = \{(1,0),(0,1)\}$ and $\beta = \{(2,1),(-3,4)\}$ are bases of $\mathbb{R}^2$. Find the matrices $T_{\alpha,\alpha}$, $T_{\alpha,\beta}$, $T_{\beta,\alpha}$, and $T_{\beta,\beta}$.

(ii) Consider the matrix:

$$A = \begin{bmatrix} 6 & 4 \\ -6 & -4 \end{bmatrix}$$

Find a general formula for the entries of A^n.

[Hint: Diagonalize A]

[100 marks]

4. (a) *Diberi subruang* $W = L(u_1, u_2 u_3)$ *yang mana*

$$u_1 = (1,1,0,0),\ u_2 = (2,-1,0,0),\ u_3 = (3,-3,0,-2)$$

Cari asas ortonormal bagi W menggunakan proses Gram-Schmidt.

(b) (i) *Biar U suatu subruang dari ruang vektor V. Jika* $U^{\perp}$ *menandakan pelengkap berortogon bagi U, tunjukkan bahawa kes yang mana suatu vektor x berada dalam kedua-dua W dan* $W^{\perp}$ *ialah hanya apabila* $x = \underline{0}$.

(ii) *Tunjukkan bahawa jika W ialah suatu subruang* $\mathbb{R}^n$ *maka* $W^{\perp} = \mathbb{R}^n$ *jika dan hanya jika* $W = \{\underline{0}\}$.

(c) *Cari garislurus padanan terbaik bagi titik-titik*

$$(-2,1), (-1,3), (0,2), (1,3), (2,1).$$

(d) (i) *Biar* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *suatu transformasi linear yang tertakrif dengan* $(x,y)T = (x-2y, -y)$ *yang mana* $\alpha = \{(1,0),(0,1)\}$ *dan* $\beta = \{(2,1),(-3,4)\}$ *adalah asas* $\mathbb{R}^2$. *Cari matriks* $T_{\alpha,\alpha}$, $T_{\alpha,\beta}$, $T_{\beta,\alpha}$, and $T_{\beta,\beta}$.

(ii) *Pertimbangkan matriks:*

$$A = \begin{bmatrix} 6 & 4 \\ -6 & -4 \end{bmatrix}$$

Cari formula umum untuk pemasukan-pemasukan dari A^n.

[Petunjuk: Pepenjurukan A]

[100 markah]

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