UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2008/2009

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MST 565 – Linear Models
[Model Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) For every square matrix $\mathbf{A}$, a scalar λ and a nonzero vector $\mathbf{x}$ can be found such that $\mathbf{Ax} = \lambda\mathbf{x}$ where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is an eigenvector. Then, show that

   (i) λ^2 is an eigenvalue of $\mathbf{A}^2$.

   (ii) if λ is an eigenvalue of the nonsingular matrix $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

   (b) Let $\mathbf{A}$ be nonsingular matrix of order n with elements a_{ij} that are functions of a scalar x. Show that

$$\frac{\partial \mathbf{A}^{-1}}{\partial x} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x}\mathbf{A}^{-1}$$

   (c) Consider $\mathbf{A}$ is symmetric and nonsingular matrix and is formed into partitioned matrix as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}'_{12} & a_{22} \end{pmatrix}.$$

   Provided that $\mathbf{A}_{11}$ is a square matrix, a_{22} is a scalar, and $\mathbf{a}_{12}$ is a vector, then, if $\mathbf{A}_{11}^{-1}$ exists, show that

$$\mathbf{A}^{-1} = \frac{1}{b}\begin{pmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}'_{12}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ -\mathbf{a}'_{12}\mathbf{A}_{11}^{-1} & 1 \end{pmatrix}$$

   where $b = a_{22} - \mathbf{a}'_{12}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$.

[25 marks]

2. (a) Let $\mathbf{y} = (y_1, y_2, y_3)'$ be a random vector with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}$$

   (i) Determine which variables are independent?

   (ii) Let $z_1 = y_1 + y_2 + y_3$ and $z_2 = 3y_1 + y_2 - 2y_3$. Find $E(\mathbf{z})$ and $\text{cov}(\mathbf{z})$ where $\mathbf{z} = (z_1, z_2)'$.

   (iii) Define $\mathbf{w} = (w_1, w_2, w_3)'$ as follows:

$$w_1 = 2y_1 - y_2 + y_3$$
$$w_2 = y_1 + 2y_2 - 3y_3$$
$$w_3 = y_1 + y_2 + 2y_3$$

   Using $\mathbf{z}$ as defined in (ii), find $\text{cov}(\mathbf{z}, \mathbf{w})$.

1. (a) *Bagi setiap matriks segiempat sama* $\mathbf{A}$, *suatu skalar* λ *dan vektor bukan sifar* $\mathbf{x}$ *boleh diperoleh supaya* $\mathbf{Ax} = \lambda\mathbf{x}$ *yang mana* λ *ialah nilai eigen untuk* $\mathbf{A}$ *dan* $\mathbf{x}$ *merupakan vektor eigen. Maka, tunjukkan bahawa*

   (i) λ^2 *merupakan nilai eigen bagi* $\mathbf{A}^2$.

   (ii) *Jika* λ *suatu nilai eigen bagi matriks tak singular* $\mathbf{A}$, *maka* $\frac{1}{\lambda}$ *merupakan nilai eigen bagi* $\mathbf{A}^{-1}$.

   (b) *Biarkan* $\mathbf{A}$ *sebagai matriks tak singular dengan peringkat* n *dengan unsur-unsur* a_{ij} *yang merupakan suatu fungsi skalar* x. *Tunjukkan bahawa:*

$$\frac{\partial \mathbf{A}^{-1}}{\partial x} = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x}\mathbf{A}^{-1}$$

   (c) *Pertimbangkan* $\mathbf{A}$ *suatu matriks simetri dan tak singular, dan dibentuk kepada Matriks terpetak sebagai*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}'_{12} & a_{22} \end{pmatrix}.$$

Diberikan $\mathbf{A}_{11}$ *adalah suatu matriks segiempat sama,* a_{22} *suatu skalar, dan* $\mathbf{a}_{12}$ *suatu vektor, maka jika* $\mathbf{A}_{11}^{-1}$ *wujud, tunjukkan bahawa*

$$\mathbf{A}^{-1} = \frac{1}{b}\begin{pmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}'_{12}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ -\mathbf{a}'_{12}\mathbf{A}_{11}^{-1} & 1 \end{pmatrix}$$

yang mana $b = a_{22} - \mathbf{a}'_{12}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$.

[25 markah]

2. (a) *Biar* $\mathbf{y} = (y_1, y_2, y_3)'$ *suatu vektor rawak dengan vektor min dan matriks kovarians*

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}$$

   (i) *Tentukan pemboleh ubah yang manakah yang tak bersandar?*

   (ii) *Biar* $z_1 = y_1 + y_2 + y_3$ *dan* $z_2 = 3y_1 + y_2 - 2y_3$. *Carikan* $E(\mathbf{z})$ *and* $\text{cov}(\mathbf{z})$ *yang mana* $\mathbf{z} = (z_1, z_2)'$.

   (iii) *Takrifkan* $\mathbf{w} = (w_1, w_2, w_3)'$ *seperti berikut:*

$$w_1 = 2y_1 - y_2 + y_3$$
$$w_2 = y_1 + 2y_2 - 3y_3$$
$$w_3 = y_1 + y_2 + 2y_3$$

Menggunakan $\mathbf{z}$ *seperti diberikan dalam (ii), cari* $\text{cov}(\mathbf{z}, \mathbf{w})$.

(b) Let random vector $\mathbf{v}$ be $N_4(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given as:

$$\boldsymbol{\mu}=\begin{pmatrix}2\\5\\-2\\1\end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix}9&0&3&3\\0&1&-1&2\\3&-1&6&-3\\3&2&-3&7\end{pmatrix}$$

If $\mathbf{v}$ is partitioned as $\mathbf{v}=(y_1,y_2,x_1,x_2)'$,

(i) What are the partitioned form for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

(ii) Find $\boldsymbol{\Sigma}_{y\cdot x}$ and $\mathbf{D}_{y\cdot x}$.

(iii) Now, find matrix of partial correlation, $\mathbf{P}_{y\cdot x}$

(c) If $\mathbf{y}$ is $N_3(\boldsymbol{\mu},\boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu}=\begin{pmatrix}3\\-2\\1\end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix}2&0&0\\0&4&0\\0&0&3\end{pmatrix},.$$

Find a symmetric matrix $\mathbf{A}$ such that $\mathbf{y}'\mathbf{A}\mathbf{y}$ is $\chi^2\left(3,\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}\right)$. What is the value $\lambda=\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$?

[30 marks]

3. (a) Suppose we use the model $y_i=\beta_0^*+\beta_1^*x_i+\varepsilon_i^*$ when the true model is

$$y_i=\beta_0+\beta_1x_i+\beta_2x_i^2+\beta_3x_i^3+\varepsilon_i$$

with $\operatorname{cov}(\mathbf{y})=\sigma^2\mathbf{I}$.

(i) Find $E(\hat{\beta}_0^*)$ and $E(\hat{\beta}_1^*)$ if observations are taken at $x=-3,-2,-1,0,1,2,3$.

(ii) Find the expected value of the variance estimator, $E(s_1^2)$ for the same values of x.

(b) In an effort to obtain maximum yield in a chemical reaction, the value of the following variables were chosen by the experimenter:

x_1 = temperature (^0C),
x_2 = concentration of a reagent (%)
x_3 = time of reaction (hours).

(b) *Biar vektor rawak* $\mathbf{v}$ *sebagai* $N_4(\boldsymbol{\mu},\boldsymbol{\Sigma})$, *yang mana* $\boldsymbol{\mu}$ *dan* $\boldsymbol{\Sigma}$ *diberikan seperti berikut:*

$$\boldsymbol{\mu}=\begin{pmatrix}2\\5\\-2\\1\end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix}9&0&3&3\\0&1&-1&2\\3&-1&6&-3\\3&2&-3&7\end{pmatrix}$$

Jika $\mathbf{v}$ *terpetak sebagai* $\mathbf{v}=(y_1,y_2,x_1,x_2)'$,

(i) *Apakah bentuk terpetak bagi* $\boldsymbol{\mu}$ *dan* $\boldsymbol{\Sigma}$ *?*

(ii) *Cari* $\boldsymbol{\Sigma}_{y\cdot x}$ *dan* $\mathbf{D}_{y\cdot x}$.

(iii) *Sekarang, cari matriks korelasi separa,* $\mathbf{P}_{y\cdot x}$.

(c) *Jika* $\mathbf{y}$ *merupakan* $N_3(\boldsymbol{\mu},\boldsymbol{\Sigma})$, *yang mana*

$$\boldsymbol{\mu}=\begin{pmatrix}3\\-2\\1\end{pmatrix},\ \boldsymbol{\Sigma}=\begin{pmatrix}2&0&0\\0&4&0\\0&0&3\end{pmatrix},.$$

Cari matriks simetri $\mathbf{A}$ *supaya* $\mathbf{y}'\mathbf{A}\mathbf{y}$ *merupakan* $\chi^2\left(3,\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}\right)$. *Apakah nilai* $\lambda=\frac{1}{2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ *?*

[30 markah]

3. (a) *Katakan kita menggunakan model* $y_i=\beta_0^*+\beta_1^*x_i+\varepsilon_i^*$ *sedangkan model yang sebenar adalah*

$$y_i=\beta_0+\beta_1x_i+\beta_2x_i^2+\beta_3x_i^3+\varepsilon_i$$

dengan $\operatorname{cov}(\mathbf{y})=\sigma^2\mathbf{I}$.

(i) *Cari* $E(\hat{\beta}_0^*)$ *dan* $E(\hat{\beta}_1^*)$ *jika cerapan dibuat pada* $x=-3,-2,-1,0,1,2,3$.

(ii) *Cari nilai jangkaan bagi penganggar varians,* $E(s_1^2)$ *untuk nilai* x *yang sama.*

(b) *Dalam usaha untuk mendapatkan hasil yang maksimum dalam suatu tindakbalas kimia, nilai pemboleh ubah berikut telah dipilih oleh seorang pengkaji:*

x_1 = *suhu* (^{o}C),
x_2 = *kepekatan reagen (%)*
x_3 = *masa tindakbalas (hours).*

Two different response variables were observed:

y_1 = percent of unchanged starting material,

y_2 = percent converted to the desired product.

The data are in Table 1.

(i) Find an estimate of $\text{cov}(\hat{\boldsymbol{\beta}})$.

(ii) Find R^2 and R_a^2.

[20 marks]

4. (a) Consider the dependent variable y_1 in the chemical reaction data in Table 1. For the model $y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$,

(i) Test the hypothesis $H_0 : \mathbf{C\beta} = \mathbf{0}$ versus $H_1 : \mathbf{C\beta} \neq \mathbf{0}$

(ii) Compute a 95% confidence interval for each β_j, using y_2 as the dependent variable in the chemical reaction data.

(b) A researcher has developed two chemical additives for increasing the mileage of gasoline. To formulate the model, he might start with the notion that without additives, a gallon yields an average of μ miles. Then if chemical 1 is added, the mileage is expected to increase by τ_1 miles per gallon, and if chemical 2 is added, the mileage would increase by τ_2 miles per gallon.

Consider the model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, $i = 1,2,$ $j = 1,2,3$ for the experiment consists of filling the tanks of six identical cars with gas then adding chemical 1 to three tanks and chemical 2 to the other three tanks.

(i) What are the matrix $\mathbf{X}$ and the vector $\boldsymbol{\beta}$ in this model.

(ii) Find $\mathbf{X'X}, (\mathbf{X'X})^{-}$ and $\mathbf{X'y}$

(iii) Find $E(\hat{\boldsymbol{\beta}})$

[25 marks]

Dua pemboleh ubah respon yang berbeza yang diperhatikan ialah:

y_1 = *peratusan bahan asal yang tidak berubah,*
y_2 = *peratusan bahan yang terbentuk.*

Data seperti yang ditunjukkan dalam Jadual 1.

(i) *Cari anggaran untuk* $\mathrm{cov}(\hat{\boldsymbol{\beta}})$.

(ii) *Cari* R^2 *dan* R_a^2.

[20 markah]

4. (a) *Pertimbangkan pemboleh ubah bersandar* y_1 *dalam data tindak balas kimia dalam Jadual 1. Untuk model*

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon ,$$

(i) *Uji hipotesis* $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ *versus* $H_1 : \mathbf{C}\boldsymbol{\beta} \neq \mathbf{0}$.

(ii) *Dapatkan selang keyakinan 95% bagi setiap* β_j, *dengan menggunakan* y_2 *sebagai pemboleh ubah bersandar dalam data tindakbalas kimia tersebut.*

(b) *Seorang penyelidik telah mencipta dua bahan tambahan kimia untuk menambah jarak perjalanan suatu gasolin. Untuk pembentukan model, beliau memulakan dengan penyataan bahawa tanpa bahan tambahan, satu gelen menghasilkan jarak purata* μ *batu. Kemudian, apabila bahan kimia 1 ditambah, jarak dijangka bertambah sebanyak* τ_1 *batu per gelen, dan jika bahan kimia 2 yang ditambah, jarak akan meningkat sebanyak* τ_2 *batu per gelen.*

Pertimbangkan model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, $i = 1,2$, $j = 1,2,3$ *bagi satu ujikaji dengan mengisi tangki minyak enam buah kereta yang serupa dengan gasoline yang ditambah dengan bahan tambahan kimia 1 kepada tiga buah kereta dan kimia 2 pada tiga buah kereta yang lain.*

(i) *Apakah matriks* $\mathbf{X}$ *dan vektor* $\boldsymbol{\beta}$ *untuk model ini.*

(ii) *Tentukan* $\mathbf{X'X}, (\mathbf{X'X})^-$ *dan* $\mathbf{X'y}$

(iii) *Dapatkan* $E(\hat{\boldsymbol{\beta}})$.

[25 markah]

Table 1. Chemical Reaction Data

y_1	y_2	x_1	x_2	x_3
41.5	45.9	162	23	3
33.8	53.3	162	23	8
27.7	57.5	162	30	5
21.7	58.8	162	30	8
19.9	60.6	172	25	5
15.0	58.8	172	25	8
12.2	58.6	172	30	5
4.3	52.4	172	30	8
19.3	56.9	167	27.5	6.5
6.4	55.4	177	27.5	6.5
37.6	46.9	157	27.5	6.5
18.0	57.3	167	32.5	6.5
26.3	55.0	167	22.5	6.5
9.9	58.9	167	27.5	9.5
25.0	50.3	167	27.5	3.5
14.1	61.1	177	20	6.5
15.2	62.9	177	20	6.5
15.9	60.0	160	34	7.5
19.6	60.6	160	34	7.5

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