UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2008/2009

April/May 2009

MGM 562 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

*[**Arahan:** Jawab **semua sepuluh** [10] soalan.]*

1. A box contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the box, c additional balls of the same color are put in with it. Now, suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $\dfrac{b}{b+r+c}$.

[10 marks]

2. Consider two boxes, one containing one black and one white marble, the other, two blacks and one white marble. A box is selected at random and a marble is drawn from the selected box.
   (a) What is the probability that the marble is black?
   (b) What is the probability that the first box was the one selected given the marble is white?

[10 marks]

3. Let the density for the continuous random variable X be given by

$$f(x) = \begin{cases} \frac{1}{2}e^{-x} & x \geq 0 \\ \frac{1}{2}e^{x} & x < 0 \end{cases}$$

   (a) Show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
   (b) Show that

$$M_X(t) = \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) \qquad -1 < t < 1$$

   (c) Use $M_X(t)$ to find $E(X)$.

[10 marks]

4. Let the probability density function of X be given by

$$f_X(x) = \begin{cases} c(4x^2 - 2x^3) & 0 < x < 2 \\ 0 & otherwise \end{cases}$$

What is $\Pr\left(\frac{1}{2} < X < \frac{3}{2}\right)$?

[10 marks]

1. *Suatu kotak mengandungi b bola hitam dan r bola merah. Satu daripada bola di dalam kotak dipilih secara rawak, tetapi apabila dimasukkan semula ke dalam kotak itu, sebanyak c bola yang sama warna dengan yang diambil, ditambahkan ke dalam kotak tersebut. Sekarang, kita memilih semula sebiji lagi bola secara rawak dari dalam kotak tersebut. Tunjukkan kebarangkalian bahawa bola pertama adalah hitam bersyaratkan bola kedua adalah merah, ialah* $\dfrac{b}{b+r+c}$.

[10 markah]

2. *Pertimbangkan dua kotak, yang pertama mengandungi satu guli hitam dan satu guli putih, dan kotak kedua mengandungi dua guli hitam dan satu guli putih. Salah satu kotak dipilih dan satu guli dipilih secara rawak daripada kotak itu.*
   (a) Apakah kebarangkalian bahawa guli itu hitam?
   (b) Apakah kebarangkalian bahawa kotak pertama terpilih diberi bahawa guli yang terpilih adalah putih?

[10 markah]

3. *Diberi taburan untuk pembolehubah selanjar,* X *adalah*

$$f(x)=\begin{cases}\dfrac{1}{2}e^{-x} & x\geq 0\\[2mm] \dfrac{1}{2}e^{x} & x<0\end{cases}$$

   (a) Tunjukkan bahawa $\int_{-\infty}^{\infty} f(x)\,dx=1$.

   (b) Tunjukkan bahawa

$$M_X(t)=\frac{1}{2}\left(\frac{1}{t+1}-\frac{1}{t-1}\right) \qquad -1<t<1$$

   (c) Dengan menggunakan $M_X(t)$, *cari* $E(X)$.

[10 markah]

4. *Diberi fungsi ketumpatan kebarangkalian bagi* X *sebagai*

$$f_X(x)=\begin{cases}c\left(4x^2-2x^3\right) & 0<x<2\\ 0 & \textit{selainnya}\end{cases}$$

Apakah $\Pr\left(\dfrac{1}{2}<X<\dfrac{3}{2}\right)$?

[10 markah]

5. Suppose the density of X is given by

$$f_X(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & otherwise \end{cases}$$

Calculate the moment generating function, $E(X)$ and $Var(X)$. Then, calculate $E\left(5X + 4X^2\right)$.

[10 marks]

6. The joint density of X and Y is given by

$$f(x, y) = \left(\frac{y^2 - x^2}{8}\right) e^{-y}, \qquad 0 < y < \infty, \qquad -y < x \leq y$$

Show that $\int_0^\infty \int_{-y}^{y} \left[\left(\frac{y^2 - x^2}{8}\right) e^{-y}\right] dxdy = 1$. Also, show that $E(X \mid Y = y) = 0$.

[10 marks]

7. Consider the following questions:

(a) Let X be exponential with mean $\frac{1}{\lambda}$, that is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Find $E(X \mid X > 1)$

(b) Let X be uniform over (0,1). Find $E(X \mid X < 1/2)$.

[10 marks]

8. In a study case over a certain period, a small firm is suffering a loss following an exponential distribution, that is, $X \sim \exp(\lambda = 0.0001)$. The probability density function (PDF) of loss is shown as follows

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

(a) Find $\Pr(7500 < X < 12000)$ using directly from the above PDF.

(b) By using Normal approximation, compute $\Pr(7500 < X < 12000)$.

[10 marks]

5. *Diberi taburan bagi* X *sebagai*

$$f_X(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \textit{selainnya} \end{cases}$$

Kira fungsi penjana momen, $E(X)$ *dan* $Var(X)$. *Seterusnya, kira* $E\left(5X + 4X^2\right)$

[10 markah]

6. *Taburan gabungan bagi* X *and* Y *diberikan seperti berikut*

$$f(x,y) = \left(\frac{y^2 - x^2}{8}\right) e^{-y}, \qquad 0 < y < \infty, \qquad -y < x \le y$$

Tunjukkan bahawa $\int_0^{\infty} \int_{-y}^{y} \left[\left(\frac{y^2 - x^2}{8}\right) e^{-y}\right] dx dy = 1$. *Juga, buktikan bahawa* $E(X \mid Y = y) = 0$.

[10 markah]

7. *Pertimbangkan soalan-soalan berikut:*

   (a) *Diberi* X *adalah taburan eksponen dengan jangkaan* $\frac{1}{\lambda}$, *iaitu*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & \textit{selainnya} \end{cases}$$

   Cari $E(X \mid X > 1)$

   (b) *Diberi* X *adalah taburan seragam pada (0,1). Cari* $E(X \mid X < 1/2)$.

[10 markah]

8. *Dalam suatu kes kajian pada suatu masa tertentu, sebuah firma kecil sedang mengalami suatu kerugian mengikuti taburan eksponen, iaitu,* $X \sim \exp(\lambda = 0.0001)$. *Taburan kerugian ditunjukkan seperti berikut*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & \textit{selainnya} \end{cases}$$

   (a) *Cari* $\Pr(7500 < X < 12000)$ *dengan menggunakan perkiraan secara terus daripada taburan di atas.*

   (b) *Dengan menggunakan taburan hampiran Normal, kira* $\Pr(7500 < X < 12000)$.

[10 markah]

9. A manuscript is sent to a typing firm consisting of typists A, B and C. If it is typed by A, then the number of errors made is a Poisson random variable with mean 2.6; if typed by B, then the number of errors is a Poisson random variable with mean 3; and if typed by C, then it is a Poisson random variable with mean 3.4. Let X denote the number of errors in the typed manuscript. Assume that each typist is equally likely to do the work. Find $E(X)$ and $Var(X)$.

[10 marks]

10. For discrete numbers of x and y, such that, $1 \le x \le y$ and $y = 1,2,3$ the probability mass function of joint distribution of X and Y is given as

$$f_{XY}(x, y) = cx$$

(a) Find c.
(b) Compute $f(x)$ and $f(y)$ respectively for $x = 1,2,3$ and $y = 1,2,3$
(c) Find $E(X)$, $Var(X)$, $E(Y)$ and $Var(Y)$

[10 marks]

9. *Suatu manuskrip dihantar kepada sebuah firma perkhidmatan menaip, yang mempunyai tukang taip A, B dan C. Jika manuskrip itu ditaip oleh A, nombor kesalahan menaip tertabur secara Poisson dengan jangkaan 2.6; jika ia ditaip oleh B, nombor kesalahan adalah pembolehubah rawak Poisson dengan jangkaan 3; dan jika ditaip oleh C, nombor kesalahan adalah pembolehubah rawak Poisson dengan jangkaan 3.4. Katakan X adalah nombor kesalahan menaip bagi manuskrip tersebut. Andaikan bahawa setiap penaip adalah sama dalam menjalankan kerja mereka. Cari $E(X)$ dan $Var(X)$.*

[10 markah]

10. *Untuk nombor-nombor diskret x dan y yang mana $1 \le x \le y$ dan $y = 1,2,3$, fungsi jisim kebarangkalian bagi gabungan X dan Y dinyatakan seperti berikut*

$$f_{XY}(x,y) = cx$$

(a) *Cari c.*
(b) *Kira $f(x)$ dan $f(y)$ masing-masing untuk $x = 1,2,3$ dan $y = 1,2,3$*
(c) *Cari $E(X)$, $Var(X)$, $E(Y)$ dan $Var(Y)$.*

[10 markah]

APPENDIX – FORMULA

(i) $\frac{d}{dx}f(x)=f'(x)$

(ii) $\frac{d}{dx}\left[a(x)b(x)\right]=a'(x)b(x)+a(x)b'(x)$

(iii) $\int a'(x)b(x)dx=a(x)b(x)-\int a(x)b'(x)dx+c$

(iv) $\frac{d}{dx}\ln(x)=\frac{1}{x}$, then, $\int\frac{1}{x}dx=\ln(x)+c$

(v) $\frac{d}{dx}\ln\left[f(x)\right]=\frac{f'(x)}{f(x)}$, then $\int\frac{f'(x)}{f(x)}dx=\ln\left[f(x)\right]+c$

(vi) $\frac{d}{dx}\left[f(x)\right]^n=nf(x)^{n-1}f'(x)$

(vii) If $X\sim\exp(\lambda)$, then

a. $f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x>0\\ 0 & otherwise\end{cases}$

b. $E(X)=\frac{1}{\lambda}$

c. $Var(X)=\frac{1}{\lambda^2}$

d. $M_X(t)=\frac{\lambda}{\lambda-t}$

(viii) If $X\sim U(a,b)$

a. $f_X(x)=\begin{cases}\frac{1}{b-a} & a<x<b\\ 0 & otherwise\end{cases}$

b. $E(X)=\frac{a+b}{2}$

c. $Var(X)=\frac{(a-b)^2}{12}$

d. $M_X(t)=\frac{e^{tb}-e^{ta}}{t(b-a)}$

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