UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2008/2009

Jun 2009

MAT 517 – Computational Linear Algebra and Function Approximation

[Aljabar Linear Pengkomputeran dan Penghampiran Fungsi]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Please use two **separate** answer books for **Section A** and **Section B**, respectively. Answer **all two** **[2]** questions from Section A and **two** **[2]** questions from Section B.

*[**Arahan:** Sila guna 2 buku jawapan yang **berbeza**, masing-masingnya untuk **Bahagian A** dan **Bahagian B**. Jawab **semua dua** **[2]** soalan daripada Bahagian A dan **dua** **[2]** soalan daripada Bahagian B.]*

Section A

1. (a) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and let $\mathbf{x} \in \mathbb{R}^n$. How many scalar additions and multiplications are necessary to compute

   (i) $(\mathbf{AB})\mathbf{x}$?

   (ii) $\mathbf{A}(\mathbf{Bx})$?

   (iii) Based on your answers in parts i) and ii), which computation is more efficient? Why?

   (b) Let $\mathbf{Q} \in \mathbb{R}^{n \times n}$ and $\mathbf{u} \in \mathbb{R}^n$ are such that $\mathbf{Q} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$.

   (i) Show that $\mathbf{Q}$ is an elementary orthogonal matrix.

   (ii) Let $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Show that, when $\mathbf{A}$ is not orthogonal, computing $\mathbf{Qx}$ is more efficient than computing $\mathbf{Ax}$, and, computing $\mathbf{QB}$ is more efficient than computing $\mathbf{AB}$.

[100 marks]

2. Consider the problem of finding a vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\|\mathbf{b} - \mathbf{Ax}\|_2^2$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \geq n$, and, $\mathbf{b} \in \mathbb{R}^m$.

   (a) Describe a numerical method for finding $\mathbf{x}$ in the case $\mathbf{A}$ is full rank.

   (b) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}.$$

   (i) Find the singular value decomposition (SVD) of $\mathbf{A}$. Hence, verify that $\mathbf{A}$ is rank deficient.

   (ii) Suppose the SVD of $\mathbf{A}$ is such that $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$. It can be shown that a vector $\mathbf{x} \in \mathbb{R}^2$ that minimizes $\|\mathbf{b} - \mathbf{Ax}\|_2^2$ is of the form $\mathbf{x} = \mathbf{Vy}$, where $\mathbf{y} = \Sigma^+\mathbf{U}^T\mathbf{b}$, and $\Sigma^+ = \left(\Sigma^T\Sigma\right)^{-1}\Sigma^T$. Find this $\mathbf{x}$ and provide a geometrical interpretation of the least squares solution.

   (iii) Are there any other nontrivial solutions to the least squares problem $\mathbf{Ax} = \mathbf{b}$? Why?

[100 marks]

Bahagian A

1. (a) *Biar* $\mathbf{A}$ *dan* $\mathbf{B}$ *menjadi matriks* $n \times n$ *dan* $\mathbf{x} \in \mathbb{R}^n$. *Berapakah bilangan hasil tambah dan hasil darab skalar yang diperlukan untuk mengira*
   (i) $(\mathbf{AB})\mathbf{x}$?
   (ii) $\mathbf{A}(\mathbf{Bx})$?
   (iii) *Berdasarkan jawapan anda di bahagian i) dan ii), pengiraan manakah yang lebih efisien? Kenapa?*

   (b) *Biar* $\mathbf{Q} \in \mathbb{R}^{n\times n}$ *dan* $\mathbf{u} \in \mathbb{R}^n$ *sedemikian rupa sehinggakan* $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T$.
   (i) *Tunjukkan bahawa* $\boldsymbol{Q}$ *ialah matrix berortogon asas.*
   (ii) *Biar* $\mathbf{x} \in \mathbb{R}^n$, *dan* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. *Tunjukkan bahawa, bila* $\mathbf{A}$ tidak *berortogon, pengiraan* $\boldsymbol{Qx}$ *adalah lebih efisyen berbanding pengiraan* $\mathbf{Ax}$, *dan, pengiraan* $\mathbf{QB}$ *adalah lebih efisyen berbanding pengiraan* $\mathbf{AB}$.

[100 markah]

2. *Pertimbangkan masalah mencari vektor* $\boldsymbol{x} \in \mathbb{R}^n$ *yang meminimumkan* $\|\boldsymbol{b} - \boldsymbol{Ax}\|_2^2$, *dengan* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $m \geq n$ *dan* $\boldsymbol{b} \in \mathbb{R}^m$.

   (a) *Huraikan* ***SATU*** *kaedah berangka untuk mencari* $\mathbf{x}$ *dalam kes* $\boldsymbol{A}$ *mempunyai pangkat penuh.*

   (b) *Biar*

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \quad \textit{dan} \quad \boldsymbol{b} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}.$$

   (i) *Cari penghuraian nilai singular (SVD) bagi A. Oleh yang demikian, tentusahkan bahawa* $\boldsymbol{A}$ *mempunyai pangkat kurang.*
   (ii) *Katakan SVD* $\boldsymbol{A}$ *ialah sedemikian rupa sehinggakan* $\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$. *Boleh ditunjukkan bahawa vektor* $\boldsymbol{x} \in \mathbb{R}^2$ *yang meminimumkan* $\|\boldsymbol{b} - \boldsymbol{Ax}\|_2^2$ *ialah dalam bentuk* $\boldsymbol{x} = \boldsymbol{Vy}$, *di mana* $\boldsymbol{y} = \Sigma^+\boldsymbol{U}^T\boldsymbol{b}$,, *dan* $\Sigma^+ = \left(\Sigma^T\Sigma\right)^{-1}\Sigma^T$. *Cari* $\mathbf{x}$ *ini dan berikan penterjemahan bergeometri terhadap penyelesaian kuasa dua terkecil.*
   (iii) *Adakah terdapat penyelesaian-penyelesaian tidak remeh yang lain bagi masalah kuasa dua terkecil* $\boldsymbol{Ax} = \boldsymbol{b}$? *Kenapa?*

[100 markah]

Section B

3. (a) If $x_0, x_1, \cdots, x_n$ are $n + 1$ distinct nodes and f is a function whose values are given at these nodes, find a unique n^{th} Lagrange interpolating polynomial $P(x)$ of degree at most n with $f(x_k) = P(x_k)$ for each $k = 0,1,\cdots,n$.

(b) Use appropriate Lagrange interpolating polynomial of degree two to approximate $f(0.9)$ if
$f(0.7) = 0.01375227, f(0.8) = 0.22363362$ and $f(1.0) = 0.65809197$.

(c) A clamped cubic spline S for a function f is defined on $[1,3]$ by

$$S(x) = \begin{cases} 3(x-1)+2(x-1)^2-(x-1)^3, & \text{for } 1 \le x < 2, \\ a+b(x-2)+c(x-2)^2+d(x-2)^3, & \text{for } 2 \le x \le 3. \end{cases}$$

If $f'(1) = f'(3)$, find a, b, c and d.

[100 marks]

4. (a) State the normal equations which are obtained when fitting the best least squares line, $y = a_0x + a_1$ to the following data:

	x_i	y_i	x_i^2	x_iy_i
	1	1.3	1	1.3
	2	3.5	4	7.0
	3	4.2	9	12.6
	4	5.0	16	20.0
	5	7.0	25	35.0
	6	8.8	36	52.8
	7	10.1	49	70.7
	8	12.5	64	100.0
	9	13.0	81	117.0
	10	15.6	100	156.0
$\sum_i$	55	81.0	385	572.4

Thus, find the linear least squares polynomial fitting these data.

(b) For $x \in [-1,1]$, the Chebyshev polynomials $\{T_n(x)\}$ are defined as $T_n(x) = \cos(n \arccos x)$, for each $n \ge 0$. Using an appropriate change of variable, show that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \ge 1$.

Bahagian B

3. (a) *Jika* $x_0, x_1, \cdots, x_n$ *adalah n + 1 nod berbeza dan f adalah suatu fungsi dengan nilai-nilai diberikan pada nod tersebut, dapatkan polinomial penginterpolasi Lagrange ke n, P(x), yang unik berdarjah paling tinggi n dengan* $f(x_k) = P(x_k)$ *bagi setiap* $k = 0, 1, \cdots, n$.

(b) *Guna polinomial penginterpolasi Lagrange berdarjah dua yang sesuai untuk menganggar* $f(0.9)$ *jika*
$f(0.7) = 0.01375227, f(0.8) = 0.22363362$ *dan* $f(1.0) = 0.65809197$.

(c) *Splin kubik pengapit S untuk fungsi f ditakrif pada* $[1,3]$ *oleh*

$$S(x) = \begin{cases} 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{for } 1 \le x < 2, \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{for } 2 \le x \le 3. \end{cases}$$

Jika $f'(1) = f'(3)$, *cari a, b, c dan d.*

[100 markah]

4. (a) *Nyatakan persamaan normal yang diperoleh apabila penyuaian garis kuasa dua terkecil terbaik,* $y = a_0 x + a_1$ *dilakukan terhadap data berikut:*

	x_i	y_i	x_i^2	$x_i y_i$
	1	1.3	1	1.3
	2	3.5	4	7.0
	3	4.2	9	12.6
	4	5.0	16	20.0
	5	7.0	25	35.0
	6	8.8	36	52.8
	7	10.1	49	70.7
	8	12.5	64	100.0
	9	13.0	81	117.0
	10	15.6	100	156.0
$\sum_i$	55	81.0	385	572.4

Dengan demikian, cari polinomial kuasa dua terkecil linear yang tersuai dengan data tersebut.

(b) *Untuk* $x \in [-1,1]$, *polinomial Chebyshev,* $\{T_n(x)\}$, *ditakrifkan sebagai* $T_n(x) = \cos(n \arccos x)$, *untuk setiap* $n \ge 0$. *Dengan menggunakan pertukaran pembolehubah yang sesuai, tunjukkan bahawa* $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ *bagi* $n \ge 1$.

(c) Show that the Chebyshev polynomials are orthogonal on (-1,1) with respect to the weight function $\omega(x) = (1-x^2)^{-\frac{1}{2}}$, by considering $\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx$, when $n \neq m$.

(d) If e^x on the interval $[-1,1]$ is approximated by $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$, find $P_3(x)$, a polynomial of degree 3 or less that best uniformly approximates $P_4(x)$ on $[-1,1]$.

[100 marks]

(c) *Tunjukkan bahawa polinomial Chebyshev adalah berortogon pada* (-1,1) *terhadap fungsi pemberat* $\omega(x)=(1-x^2)^{-\frac{1}{2}}$, *dengan mempertimbangkan* $\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx$, *apabila* $n\neq m$.

(d) *Jika* e^x *pada selang* $[-1,1]$ *dianggar oleh* $P_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$, *cari* $P_3(x)$, *suatu polinomial berdarjah 3 atau kurang yang menganggar* $P_4(x)$ *secara seragam terbaik pada* $[-1,1]$.

[100 markah]

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