UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2008/2009

April/May 2009

MAT 516 – Curve and Surface Methods for CAGD
[Kaedah Lengkung dan Permukaan untuk RGBK]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

1. The nth degree Bézier curve is defined as $P(t)=\sum_{i=0}^{n} V_i B_i^n(t),\quad 0\le t\le 1$ with

   $B_i^n(t)=\dfrac{n!t^i(1-t)^{n-i}}{(n-i)!i!}$ and V_i its Bézier control points.

   (a) Show that a cubic Bézier curve can be represented in a matrix form TMV

   where $T=\begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix}$, $M=\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{bmatrix}$ and $V=\begin{bmatrix}V_0\\ V_1\\ V_2\\ V_3\end{bmatrix}$.

   (b) Let $Q(t)=P(t+1)$.

   (i) Use the result in part (a) to find the matrix C so that $Q(t)=TCMV$.

   (ii) Using $M^{-1}=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & \frac{1}{3} & 0 & 0\\ 1 & \frac{2}{3} & \frac{1}{3} & 0\\ 1 & 1 & 1 & 1\end{bmatrix}$, write the control points $Q_i, i=0,1,2,3$ of $Q(t)$, in terms of V_i.

   (iii) Construct the control polygon of $Q(t)$ (and its control points), the control polygon of $P(t)$ (and its control points) and both curves in the same figure. Show that $P(t)$ and $Q(t)$ satisfy the C^1 continuity conditions at the common control point.

   (c) State the conditions and sketch the position of the control points of two adjacent cubic Bézier curves,

   $P_3(t)=\sum_{i=0}^{3} V_i B_i^3(t)$ and $Q_3(t)=\sum_{i=0}^{3} W_i B_i^3(t), 0\le t\le 1,$ which meet at the common control point with C^0, C^1, C^2, G^1 and G^2 continuity, respectively.

[100 marks]

1. *Lengkung Bézier berdarjah n ditakrif sebagai* $P(t) = \sum_{i=0}^{n} V_i B_i^n(t), \quad 0 \le t \le 1$

dengan $B_i^n(t) = \dfrac{n!t^i(1-t)^{n-i}}{(n-i)!i!}$ *dan* V_i *adalah titik kawalan Bézier.*

(a) *Tunjukkan bahawa suatu lengkung Bézier kubik boleh diwakili dalam bentuk matriks TMV dengan*

$$T = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \text{ dan } V = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

(b) *Andaikan* $Q(t) = P(t+1)$.

(i) *Guna keputusan di bahagian (a) bagi mencari matriks C supaya* $Q(t) = TCMV$.

(ii) *Dengan menggunakan* $M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, *tulis titik kawalan* $Q_i, i = 0, 1, 2, 3$ *lengkung* $Q(t)$, *dalam sebutan* V_i.

(iii) *Bina poligon kawalan lengkung* $Q(t)$ *(dan titik kawalannya), poligon kawalan lengkung* $P(t)$ *(dan titik kawalannya) dan kedua-dua lengkung dalam rajah yang sama. Tunjukkan bahawa* $P(t)$ *dan* $Q(t)$ *memenuhi syarat keselanjaran* C^1 *pada titik kawalan sepunya.*

(c) *Nyatakan syarat terhadap titik-titik kawalan dan lakar kedudukan titik-titik kawalan dua lengkung kubik Bézier bersebelahan,* $P_3(t) = \sum_{i=0}^{3} V_i B_i^3(t)$ *dan* $Q_3(t) = \sum_{i=0}^{3} W_i B_i^3(t), 0 \le t \le 1,$ *yang bertemu pada titik kawalan sepunya dengan masing-masingnya berkeselanjaran* C^0, C^1, C^2, G^1 *dan* G^2.

[100 markah]

2. A B-spline curve of order k is defined by $P(t) = \sum_{i=0}^{n} V_i N_{i,k}(t), t \in [t_{k-1}, t_{n+1}]$. $V_i, i = 0,1,\cdots,n$ are its control points, $T = \{t_0, t_1, \cdots, t_{n+k}\}$ is a knot vector $(n \geq k-1)$ and $N_{i,k}(t)$ are the normalized B-spline basis functions of order k defined recursively by

$$N_{i,1}(t) = \begin{cases} 1, \text{if } t \in [t_i, t_{i+1}) \\ 0, \text{otherwise} \end{cases}$$

and $N_{i,k}(t) = \dfrac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$, where $0 \leq i \leq n$.

(a) Explain the conditions on $\{t_0, t_1, t_2, \ldots, t_{n+k}\}$ so that the B-spline curve of order k (or degree k-1) interpolates the first and last control points.

(b) If {-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8} is the given knot vector, sketch the basis functions and state the maximum number of control points and the interval for t in order to generate a B-spline curve of order 4 (or of degree 3).

(c) A quadratic B-spline curve is defined by

$$P(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} N_{03}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} N_{13}(t) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} N_{23}(t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} N_{33}(t), \text{ for } t \in [1, 5]$$

where $N_{i3}(t), i = 0, 1, 2, 3$ are B-spline basis functions of order 3 with knots $\{0, 0, 1, 3, 5, 6, 7\}$.

(i) Find $N_{i3}(t)$, for $i = 0, 1, 2, 3$.

(ii) Show that $\sum_{i=0}^{3} N_{i3}(t) = 1$ for $t \in [1, 5]$.

(iii) Evaluate $P(1.5)$, $P(3)$ and $P(5)$.

(iv) By using knots $\{0, 0, 0, 0, 1, 1, 1, 1\}$, show that $N_{j4}(t) = B_j^3(t)$, for $j = 0, 1, 2, 3$ and $t \in [0, 1]$, where $B_j^3(t)$ is as defined in Question 1.

[100 marks]

2. *Satu lengkung splin-B peringkat k ditakrifkan oleh*

$P(t) = \sum_{i=0}^{n} V_i N_{i,k}(t), t \in [t_{k-1}, t_{n+1}]$. $V_i, i = 0,1,\cdots,n$ *adalah titik kawalannya,* $T = \{t_0, t_1, \cdots, t_{n+k}\}$ *ialah vektor simpulan* $(n \geq k-1)$ *dan* $N_{i,k}(t)$ *adalah fungsi asas splin-B ternormal peringkat k ditakrif secara rekursi oleh*

$$N_{i,1}(t) = \begin{cases} 1, \text{ jika } t \in [t_i, t_{i+1}) \\ 0, \text{ selainnya} \end{cases}$$

dan $N_{i,k}(t) = \dfrac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \dfrac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$, *dengan* $0 \leq i \leq n$.

(a) *Terangkan syarat-syarat ke atas* $\{t_0, t_1, t_2, \ldots, t_{n+k}\}$ *supaya lengkung splin-B peringkat k (atau darjah k-1) menginterpolasi titik kawalan yang pertama dan titik kawalan yang terakhir.*

(b) *Jika {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} adalah vektor simpulan, lakar fungsi asas dan nyata bilangan maksimum titik kawalan dan selang bagi t untuk menjana lengkung splin-B peringkat 4 (atau darjah 3).*

(c) *Lengkung splin-B kuadratik ditakrif oleh*

$$P(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} N_{03}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} N_{13}(t) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} N_{23}(t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} N_{33}(t), \text{ bagi } t \in [1, 5]$$

dengan $N_i^3(t), i = 0, 1, 2, 3$ *adalah fungsi asas splin-B peringkat 3 dengan simpulan* $\{0, 0, 1, 3, 5, 6, 7\}$.

(i) *Cari* $N_{i3}(t)$, *bagi* $i = 0, 1, 2, 3$.

(ii) *Tunjukkan bahawa* $\sum_{i=0}^{3} N_{i3}(t) = 1$ *bagi* $t \in [1, 5]$.

(iii) *Nilaikan* $P(1.5), P(3)$ *dan* $P(5)$.

(iv) *Dengan menggunakan simpulan* $\{0, 0, 0, 0, 1, 1, 1, 1\}$, *tunjukkan bahawa* $N_{j4}(t) = B_j^3(t)$, *bagi* $j = 0,1,2,3$ *dan* $t \in [0,1]$, *dengan* $B_j^3(t)$ *sebagai yang tertakrif di dalam Soalan 1.*

[100 markah]

3. Let

$$P_{00}=\begin{pmatrix}0\\0\\0\end{pmatrix}, P_{01}=\begin{pmatrix}0\\1\\0\end{pmatrix}, P_{02}=\begin{pmatrix}0\\2\\0\end{pmatrix}, P_{10}=\begin{pmatrix}1\\0\\0\end{pmatrix}, P_{11}=\begin{pmatrix}1\\1\\1\end{pmatrix}, P_{12}=\begin{pmatrix}1\\2\\0\end{pmatrix},$$

$$P_{20}=\begin{pmatrix}2\\0\\0\end{pmatrix}, P_{21}=\begin{pmatrix}2\\1\\1\end{pmatrix}, P_{22}=\begin{pmatrix}2\\2\\0\end{pmatrix}, P_{30}=\begin{pmatrix}3\\0\\0\end{pmatrix}, P_{31}=\begin{pmatrix}3\\1\\0\end{pmatrix}, P_{32}=\begin{pmatrix}3\\2\\0\end{pmatrix}.$$

(a) The biquadratic B-spline patch, P is defined by $P(u,v)=\frac{1}{4}UBWB^TV^T$, $0\le u,v\le 1$ where

$$U=\begin{bmatrix}u^2 & u & 1\end{bmatrix},\ V=\begin{bmatrix}v^2 & v & 1\end{bmatrix},\ B=\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{bmatrix},\ W=\begin{bmatrix}P_{00} & P_{01} & P_{02}\\ P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\end{bmatrix}.$$

(i) Write its four corners $P(0,0), P(0,1), P(1,0)$ and $P(1,1)$ as a barycentric sum of nine control points. Find them.

(ii) Calculate the midpoint $P(0.5, 0.5)$.

(b) Let Q be the adjacent patch defined by $Q(u,v)=\frac{1}{4}UBRB^TV^T$, $0\le u,v\le 1$ where

U,V and B are defined as in part (a), and $R=\begin{bmatrix}P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\\ P_{30} & P_{31} & P_{32}\end{bmatrix}$.

Show that $P(u,v)$ in part (a) and $Q(u,v)$ satisfy the C^1 continuity conditions along their common edge.

[100 marks]

3 *Andaikan*

$$P_{00}=\begin{pmatrix}0\\0\\0\end{pmatrix}, P_{01}=\begin{pmatrix}0\\1\\0\end{pmatrix}, P_{02}=\begin{pmatrix}0\\2\\0\end{pmatrix}, P_{10}=\begin{pmatrix}1\\0\\0\end{pmatrix}, P_{11}=\begin{pmatrix}1\\1\\1\end{pmatrix}, P_{12}=\begin{pmatrix}1\\2\\0\end{pmatrix},$$

$$P_{20}=\begin{pmatrix}2\\0\\0\end{pmatrix}, P_{21}=\begin{pmatrix}2\\1\\1\end{pmatrix}, P_{22}=\begin{pmatrix}2\\2\\0\end{pmatrix}, P_{30}=\begin{pmatrix}3\\0\\0\end{pmatrix}, P_{31}=\begin{pmatrix}3\\1\\0\end{pmatrix}, P_{32}=\begin{pmatrix}3\\2\\0\end{pmatrix}.$$

(a) *Tampalan splin-B bikuadratik, P ditakrifkan oleh*

$$P(u,v)=\frac{1}{4}UBWB^TV^T,\ 0\le u,v\le 1$$

dengan

$$U=\begin{bmatrix}u^2 & u & 1\end{bmatrix},\ V=\begin{bmatrix}v^2 & v & 1\end{bmatrix},\ B=\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 1 & 0\end{bmatrix},\ W=\begin{bmatrix}P_{00} & P_{01} & P_{02}\\ P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\end{bmatrix}.$$

(i) *Tulis empat penjurunya,* $P(0,0), P(0,1), P(1,0)$ *and* $P(1,1)$ *sebagai hasil tambah baripusat sembilan titik kawalan. Cari keempat-empat penjuru tersebut.*

(ii) *Kira titik tengah* $P(0.5, 0.5)$.

(b) *Andaikan Q adalah tampalan bersebelahan yang ditakrif oleh*

$$Q(u,v)=\frac{1}{4}UBRB^TV^T,\ 0\le u,v\le 1 \text{ dengan}$$

U, V *dan B tertakrif seperti di bahagian (a), dan* $R=\begin{bmatrix}P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\\ P_{30} & P_{31} & P_{32}\end{bmatrix}$.

Tunjukkan bahawa $P(u,v)$ *di bahagian (a) dan* $Q(u,v)$ *memenuhi syarat keselanjaran* C^1 *sepanjang sisi sepunya P dan Q.*

[100 markah]

- ooo O ooo -