UNIVERSITI SAINS MALAYSIA

Peperiksaan Kursus Semasa Cuti Panjang
Sidang Akademik 2008/2009

Jun 2009

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa: 3 jam]

Please check that this examination paper consists of TEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini].

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Find the value of c such that $f(x)$ is a probability mass function for each of the following section.

(i) $f(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, \ldots$, and zero elsewhere.

(ii) $f(x) = (2x+1)c$, $x = 0, 1, 2, \ldots, 6$, and zero elsewhere.

(iii) $f(x) = xc$, $x = 1, 2, \ldots, N$ and zero elsewhere.

[30 marks]

(b) Assume that the random variable X is gamma distributed, that is $X \sim G(\alpha,\lambda)$.

(i) Show that the moment generating function of X is $\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$, for $t < \lambda$ if given that the probability density function of X is

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x), \alpha > 0, \lambda > 0$$

(ii) From the moment generating function in (i), find the mean of X.

(iii) Hence, find the variance of X.

[30 marks]

(c) Let $f(x,y) = 2$, $0 < x < y < 1$ and zero elsewhere be the joint probability density function of X and Y.

(i) Find the conditional expectation of X given $Y = y$.

(ii) Find the conditional expectation of Y given $X = x$.

[40 marks]

1. (a) *Cari nilai c supaya f(x) untuk setiap bahagian yang berikut merupakan suatu fungsi jisim kebarangkalian.*

(i) $f(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, \ldots,$ *dan sifar sebaliknya.*

(ii) $f(x) = (2x + 1)c$, $x = 0, 1, 2, \ldots, 6,$ *dan sifar sebaliknya.*

(iii) $f(x) = xc$, $x = 1, 2, \ldots, N$ *dan sifar sebaliknya.*

[30 markah]

(b) *Andaikan pembolehubah rawak X bertaburan gama, iaitu* $X \sim G(\alpha, \lambda)$.

(i) *Tunjukkan bahawa fungsi penjana momen X ialah* $\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$, *bagi* $t < \lambda$ *jika diberi fungsi ketumpatan kebarangkalian X ialah*

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x), \alpha > 0, \lambda > 0$$

(ii) *Daripada fungsi penjana momen di (i), cari min X.*

(iii) *Seterusnya, cari varians X.*

[30 markah]

(c) *Biarkan* $f(x,y) = 2$, $0 < x < y < 1$ *dan sifar sebaliknya sebagai fungsi ketumpatan kebarangkalian tercantum X dan Y.*

(i) *Cari jangkaan bersyarat X diberi* $Y = y$.

(ii) *Cari jangkaan bersyarat Y diberi* $X = x$.

[40 markah]

2. (a) If $X_1, X_2, \ldots, X_n$ is a random sample from the distribution $N(\mu, 4\sigma^2)$, and $\overline{X}_m$ is defined as $\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i, \quad m \le n,$

Find the distribution of the following statistics:

(i) $m\overline{X}_m - n\overline{X}_n$

(ii) $(m-1)\overline{X}_m + (n-1)\overline{X}_n$

(iii) $\dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\displaystyle\sum_{i=1}^{m} \frac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

[40 marks]

(b) If $Y_1, Y_2, \ldots, Y_5$ are order statistics from a random sample of size 5 with density function

$$f(x) = \begin{cases} \beta e^{-\beta x}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases},$$

show that Y_3 and $Y_5 - Y_3$ are independent.

[30 marks]

(c) For a random sample of size, n, find the method of moment estimator for θ_1 and θ_2 if the random sample is taken from the distribution $G(\theta_1, \theta_2)$.

[30 marks]

2. (a) *Jika* $X_1, X_2, \ldots, X_n$ *merupakan sampel rawak daripada taburan* $N(\mu, 4\sigma^2)$, *dan* $\overline{X}_m$ *ditakrifkan sebagai,* $\overline{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i$, $m \leq n$,

Cari taburan setiap statistik berikut:

(i) $m\overline{X}_m - n\overline{X}_n$

(ii) $(m-1)\overline{X}_m + (n-1)\overline{X}_n$

(iii) $\dfrac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

(iv) $\displaystyle\sum_{i=1}^{m} \frac{m(\overline{X}_m - \mu)^2}{4\sigma^2}$

[40 markah]

(b) *Jika* $Y_1, Y_2, \ldots, Y_5$ *ialah statistik tertib daripada suatu sampel rawak saiz 5 dengan fungsi ketumpatan*

$$f(x) = \begin{cases} \beta e^{-\beta x}, & x \geq 0 \\ 0, & di\ tempat\ lain \end{cases},$$

tunjukkan bahawa Y_3 *dan* $Y_5 - Y_3$ *adalah tak bersandar.*

[30 markah]

(c) *Untuk sampel rawak saiz, n, cari penganggar kaedah momen untuk* θ_1 *dan* θ_2 *jika sampel rawak itu diambil daripada taburan* $G(\theta_1, \theta_2)$.

[30 markah]

3. (a) Based on a random sample of size n, use the factorization theorem to find a sufficient statistic for each of the following distributions:

(i) $f(x;\theta)=\frac{1}{\theta}I_{(0,\theta)}(x);\ \ \theta>0.$

(ii) $f(x;\theta)=\theta^2 xe^{-x\theta}I_{(0,\infty)}(x);\ \ \theta>0.$

[20 marks]

(b) Based on a random sample of size n, find a complete sufficient statistic for each of the following distributions:

(i) $f(x;\theta)=(\theta+1)x^{\theta}I_{(0,1)}(x);\ \ \theta>-1.$

(ii) $f(x;\theta)=\frac{\log\theta}{\theta-1}\theta^{x}I_{(0,1)}(x);\ \ \theta>1.$

[20 marks]

(c) Assume that $X_1, X_2,\dots, X_n$ is a random sample from the exponential distribution with density function

$$f(x;\theta)\ =\ \theta e^{-\theta x}I_{(0,\infty)}(x);\ \ \theta>0.$$

Find the uniformly minimum variance unbiased estimator (UMVUE) for the median of the distribution.

[30 marks]

(d) Assume that $X_1, X_2,\dots, X_n$ denotes a random sample from the $P_0(\lambda)$ distribution with density function

$$f(x;\lambda)\ =\ \frac{e^{-\lambda}\lambda^{x}}{x!},\ \ x=0,1,2,\dots$$

Find the Cramer-Rao lower bound for the variance of the unbiased estimator of $e^{-\lambda}$.

[30 marks]

3. (a) *Berdasarkan suatu sampel rawak bersaiz n,guna teorem pemfaktoran untuk mencari statistic cukup bagi setiap taburan berikut:*

(i) $f(x;\theta)=\frac{1}{\theta}I_{(0,\theta)}(x)$; $\theta > 0$.

(ii) $f(x;\theta)=\theta^2 xe^{-x\theta}I_{(0,\infty)}(x)$; $\theta > 0$.

[20 markah]

(b) *Berdasarkan suatu sampel rawak bersaiz n, cari suatu statistik cukup dan lengkap bagi setiap yang berikut:*

(i) $f(x;\theta)=(\theta+1)x^{\theta}I_{(0,1)}(x)$; $\theta > -1$.

(ii) $f(x;\theta)=\frac{\log\theta}{\theta-1}\theta^{x}I_{(0,1)}(x)$; $\theta > 1$.

[20 markah]

(c) *Andaikan $X_1, X_2,..., X_n$ sampel rawak daripada taburan eksponen yang mempunyai fungsi ketumpatan*

$$f(x;\theta) = \theta e^{-\theta x}I_{(0,\infty)}(x);\quad \theta > 0.$$

Cari penganggar saksama bervarians minimum secara seragam (PSVMS) untuk median taburan.

[30 markah]

(d) *Andaikan $X_1, X_2,..., X_n$ menandai sampel rawak daripada taburan $P_0(\lambda)$ dengan fungsi ketumpatan*

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!},\quad x = 0, 1, 2, \ldots$$

Cari batas bawah Cramer-Rao bagi varians penganggar saksama $e^{-\lambda}$.

[30 markah]

4. (a) Assume that $\overline{X}$ denotes the sample mean for a random sample of size n from a $N(\mu, 9)$ distribution. Find the value of n so that $(\overline{X}-1, \overline{X}+1)$ is a 95% confidence interval for μ.

[30 marks]

(b) Assume that $X_1, X_2, ..., X_{10}$ is a random sample of size 10 from a $N(0, \theta)$ distribution, where $\theta > 0$. For testing $H_0 : \theta = 1$ vs. $H_1 : \theta > 1$, the following critical region is used:

$$C = \left\{ (x_1, x_2, ..., x_{10}) : \sum_{i=1}^{10} x_i^2 \geq c \right\}.$$

Find c if the size of the critical region C is 0.05.

[30 marks]

(c) Assume that X is a single observation from a distribution having probability density function

$$f(x;\theta) = (1+\theta) x^{\theta} I_{(0,1)}(x); \quad \theta > -1.$$

(i) Find the most powerful test of size-α for testing $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$.

(ii) For testing $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$, the following test is used:

Reject H_0 if and only if $X \geq \dfrac{3}{4}$.

Find the power function and the size of the test.

[40 marks]

4. (a) *Andaikan bahawa* $\overline{X}$ *menandakan min sampel bagi suatu sampel rawak bersaiz n daripada taburan* $N(\mu, 9)$. *Cari nilai n supaya* $(\overline{X}-1, \overline{X}+1)$ *adalah suatu selang keyakinan 95% bagi* μ.

[30 markah]

(b) *Andaikan bahawa* $X_1, X_2, \ldots, X_{10}$ *adalah suatu sampel rawak bersaiz 10 daripada taburan* $N(0, \theta)$, *yang mana* $\theta > 0$. *Bagi menguji* $H_0 : \theta = 1$ *lawan.* $H_1 : \theta > 1$, *rantau genting berikut digunakan:*

$$C = \left\{ (x_1, x_2, \ldots, x_{10}) : \sum_{i=1}^{10} x_i^2 \geq c \right\}.$$

Cari c jika saiz rantau genting C adalah 0.05.

[30 markah]

(c) *Andaikan bahawa X adalah suatu cerapan tunggal daripada taburan yang mempunyai fungsi ketumpatan kebarangkalian*

$$f(x;\theta) = (1+\theta)x^{\theta} I_{(0,1)}(x); \quad \theta > -1.$$

(i) *Cari ujian paling berkuasa saiz* α *bagi menguji*

$$H_0 : \theta = 0 \ \ vs. \ \ H_1 : \theta = 1.$$

(ii) *Bagi menguji* $H_0 : \theta \leq 0$ *lawan.* $H_1 : \theta > 0$, *ujian berikut digunakan : Tolak* H_0 *jika dan hanya jika* $X \geq \frac{3}{4}$.

Cari fungsi kuasa dan saiz ujian ini.

[40 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\dots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x}p^x q^{n-x}\, I_{\{0,1,\dots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x I_{\{0,1,\dots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\dots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^{\alpha},\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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