UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2008/2009

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MAT 111 – Linear Algebra
[Aljabar Linear]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of NINE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan**: Jawab **semua empat** [4] soalan.]*

1. (a) (i) A square matrix A is said to be *symmetric* if $A^T = A$, where A^T is the transpose of A. Give an example of a 3×3 symmetric matrix.

(ii) A square matrix B is said to be *skew-symmetric* if $B^T = -B$. Give an example of a 3×3 skew-symmetric matrix.

(iii) Let A and B be square matrices of same size such that A is symmetric and B is skew-symmetric. Determine whether $AB - BA$ is symmetric or skew-symmetric. Then verify it using the examples given in part (i) and part (ii).

(iv) Using the example given in part (ii), compute B^2. Is B^2 symmetric, skew-symmetric or neither?

(v) Give an example of two 2×2 matrices A_1 and A_2 such that A_1 and A_2 are symmetric but A_1A_2 is not symmetric.

(b) The Universal Product Code, or UPC, is a code associated with the bar codes found on many types of merchandise. The black and white bars that are scanned by a laser at a store's checkout counter correspond to a 10-ary vector $\bar{u} = [u_1, u_2, \ldots, u_{11}, d]$ of length 12. The first 11 components form a vector that gives manufacturer and product information, the last component d is a check digit chosen so that $\bar{c}\cdot\bar{u} = 0$ in $\mathbb{Z}_{10}$, where the check vector $\bar{c}$ is the vector $[3,1,3,1,3,1,3,1,3,1,3,1]$. That is, after rearranging,

$$3(u_1+u_3+u_7+u_9+u_{11}) + (u_2+u_4+u_6+u_8+u_{10}) + d = 0.$$

In other words, the check digit is chosen so that the left-hand side of the expression is a multiple of 10. For example the UPC shown in Figure 1,

$$\begin{aligned}\bar{c}\cdot\bar{u} &= 3(0+6+0+2+1+5)+(3+0+0+9+4)+d\\ &= 3(4)+1(6)+d\\ &= 8+d.\end{aligned}$$

0 36000 29145 2

Figure 1

The check digit d then must be 2.

Consider the UPC $[0,4,6,9,5,6,1,8,2,0,1,5]$.

(i) Show that this UPC cannot be correct.

(ii) Assuming that a single error was made and that the incorrect digit is the 6 in the third entry. Find the correct UPC.

[100 marks]

1. (a) (i) *Satu matriks segiempat sama A dikatakan simetri jika $A^T = A$, dengan A^T ialah transposisi A. Beri satu contoh matriks 3×3 yang simetri.*

(ii) *Satu matriks segiempat sama B dikatakan simetri pencong jika $B^T = -B$. Beri satu contoh matriks 3×3 yang simetri pencong.*

(iii) *Andai A dan B matriks segiempat sama bersaiz sama sedemikian hingga A adalah simetri dan B adalah simetri pencong. Tentukan sama ada $AB - BA$ adalah simetri atau simetri pencong. Kemudian tentusahkan keputusan anda dengan menggunakan contoh dalam bahagian (i) dan bahagian (ii).*

(iv) *Dengan menggunakan contoh dalam bahagian (ii), kira B^2. Adakah B^2 simetri, semetri pencong atau bukan salah satu?*

(v) *Beri contoh dua matriks 2×2 A_1 dan A_2 sedemikian A_1 dan A_2 masing-masing adalah simetri tetapi $A_1 A_2$ tak simetri.*

(b) *Kod Barangan Semesta ("Universal Product Code", atau UPC) adalah sejenis kod palang yang biasa dijumpai di pelbagai barangan jualan. Palang-palang hitam dan putih yang biasa diimbas di kaunter semak dalam pasaraya sebenarnya bersepadan dengan vektor persepuluhan $\vec{u} = [u_1, u_2, \ldots, u_{11}, d]$ dengan jarak 12. 11 komponen pertama membentuk satu vektor yang memberi maklumat tentang pengeluar dan produk, komponen terakhir d ialah digit semak yang dipilih supaya $\vec{c} \cdot \vec{u} = 0$ dalam $\mathbb{Z}_{10}$, dengan vektor semak $\vec{c}$ ialah vektor $[3,1,3,1,3,1,3,1,3,1,3,1]$. Ini bermakna selepas penyusunan semula,*

$$3(u_1 + u_3 + u_7 + u_9 + u_{11}) + (u_2 + u_4 + u_6 + u_8 + u_{10}) + d = 0.$$

Dengan kata lain, digit semak adalah dipilih sedemikian hingga sebelah kiri persamaan ialah gandaan 10. Sebagai contoh UPC dalam Rajah 1,

$$\begin{aligned} \vec{c} \cdot \vec{u} &= 3(0+6+0+2+1+5) + (3+0+0+9+4) + d \\ &= 3(4) + 1(6) + d \\ &= 8 + d. \end{aligned}$$

0 36000 29145 2

Rajah 1

Maka digit semak ialah 2.

Pertimbangkan UPC $[0,4,6,9,5,6,1,8,2,0,1,5]$.

(i) *Tunjukkan bahawa UPC ini tidak betul.*

(ii) *Anggap satu ralat telah dibuat dan digit yang salah ialah 6 pada kemasukan ketiga. Cari UPC yang betul.*

[100 markah]

2. (a) Let $T : M_{2\times 2} \to \mathbb{R}$ be a linear transformation for which

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T = 1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} T = 2, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} T = 3, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} T = 4.$$

Find $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} T$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} T$.

(b) Define linear transformations $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ and $S : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ by $(a+bx)T = a + (a+b)x + 2bx^2$, $(\alpha + \beta x + \gamma x^2)S = \beta + 2\gamma x$.
Compute $(3 + 2x - x^2)(S \circ T)$. Can you compute $(a+bx)(T \circ S)$? If yes compute it.

(c) Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V and let $T : V \to W$ be a linear transformation.

(i) If $\{v_1T, v_2T, \ldots, v_nT\}$ is linearly independent in W, show that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V.

(ii) Give an example of $T : \mathbb{R}^2 \to \mathbb{R}^2$ to show that the converse in part (i) is false. That is, it is not necessarily true that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V, then $\{v_1T, v_2T, \ldots, v_nT\}$ is linearly independent in W.

[100 marks]

2. (a) *Andai* $T: M_{2\times 2} \to \mathbb{R}$ *satu transformasi linear sedemikian*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T = 1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} T = 2, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} T = 3, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} T = 4.$$

Cari $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} T$ *dan* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} T$.

(b) *Takrif transformasi linear* $T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$ *dan* $S: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ *sebagai* $(a+bx)T = a + (a+b)x + 2bx^2$, $(\alpha + \beta x + \gamma x^2)S = \beta + 2\gamma x$. *Hitung* $(3 + 2x - x^2)(S \circ T)$. *Boleh anda hitung* $(a+bx)(T \circ S)$? *Jika ya, hitungkan.*

(c) *Andai* $v_1, v_2, \ldots, v_n$ *vektor-vektor dalam suatu ruang vektor* V *dan* $T: V \to W$ *satu transformasi linear.*

(i) *Jika* $\{v_1T, v_2T, \ldots, v_nT\}$ *tak bersandar linear dalam* W, *tunjukkan bahawa* $\{v_1, v_2, \ldots, v_n\}$ *tak bersandar linear dalam* V.

(ii) *Beri satu contoh* $T: \mathbb{R}^2 \to \mathbb{R}^2$ *untuk menunjukkan bahawa akas bahagian (i) tidak benar, iaitu tidak semestinya benar bahawa jika* $\{v_1, v_2, \ldots, v_n\}$ *tak bersandar linear dalam* V, *maka* $\{v_1T, v_2T, \ldots, v_nT\}$ *tak bersandar linear dalam* W.

[100 markah]

3. (a) (i) Given that the set $S = \{(-1,-1,1),(1,1,0)\}$ is linearly independent. Show that S can be enlarged to a basis of $\mathbb{R}^3$ by adding an element of the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ to S.

(ii) Suppose U and W are subspaces of a finite dimensional vector space V. Prove that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.
[Hint: Start by letting $\{v_1, v_2, \ldots, v_m\}$ be a basis of $U \cap W$]

(b) Given a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $(x,y,z)T = (x+y+z, y+z, y+z)$.

(i) Find a basis for the kernel of T.

(ii) Find a basis for the image of T.

(iii) Verify the dimension theorem using your answers in parts (i) and (ii).

(iv) Is T one-to-one? Is T onto? Explain your answers.

(c) Find the basis of the column space of A where

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 \\ -1 & 0 & 2 & -4 & 1 \\ -1 & 4 & 0 & -4 & 3 \end{bmatrix}.$$

Then deduce the rank of A.

(d) (i) The following elementary row operations, $R_{12}(-2)$ and $R_{13}(-3)$ (Step 1), $R_{23}(-1)$ (Step 2), $R_2(\frac{1}{2})$ and $R_3(-\frac{1}{3})$ (Step 3), $R_{32}(-2)$ and $R_{31}(-2)$ (Step 4) and $R_{21}(-1)$ (Step 5) are performed to obtain the row-reduced echelon form (RREF) for the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 4 & 8 & 4 \\ 3 & 6 & 5 & 7 & 7 \end{array}\right].$$

Show the procedure in detail to obtain the solutions for the system (use the variables x, y, z and t)

(ii) A system of linear equations has the following equivalent system:

$$\begin{aligned} x + y \qquad + t &= 4 \\ -2y \quad - 6t &= -1 \\ z \quad - t &= 1 \end{aligned}$$

Show that there are infinitely many solutions to the system with the form $(\frac{7}{2}, \frac{1}{2}, 1, 0) + k(2,-3,1,1)$, $k \in \mathbb{R}$

Show also how you can verify these solutions.

[100 marks]

3. (a) (i) *Diberi set* $S = \{(-1,-1,1),(1,1,0)\}$ *adalah tak bersandar linear. Tunjukkan bahawa* S *dapat diperkembangkan menjadi satu asas untuk* $\mathbb{R}^3$ *dengan menambah satu elemen daripada asas piawai* $\{(1,0,0),(0,1,0),(0,0,1)\}$ *kepada* S.

(ii) *Andaikan* U *dan* W *subruang untuk ruang vektor bermatra terhingga* V. *Buktikan bahawa*

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

[Petunjuk: Mula dengan mempertimbangkan $\{v_1, v_2, \ldots, v_m\}$ *sebagai asas untuk* $U \cap W$ *]*

(b) *Satu transformasi linear* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *ditakrif sebagai* $(x,y,z)T = (x+y+z, y+z, y+z)$.

(i) *Cari asas untuk inti bagi* T.

(ii) *Cari asas untuk imej bagi* T.

(iii) *Tentusahkan teorem matra dengan menggunakan jawapan anda dalam bahagian (i) dan (ii).*

(iv) *Adakah* T *satu dengan satu? Adakah* T *keseluruh? Jelaskan jawapan anda.*

(c) *Cari asas untuk ruang lajur bagi* A *dengan*

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 \\ -1 & 0 & 2 & -4 & 1 \\ -1 & 4 & 0 & -4 & 3 \end{bmatrix}.$$

Kemudian deduksikan pangkat untuk A.

(d) (i) *Operasi baris permulaan berikut,* $R_{12}(-2)$ *dan* $R_{13}(-3)$ *(Langkah 1),* $R_{23}(-1)$ *(Langkah 2),* $R_2(\frac{1}{2})$ *and* $R_3(-\frac{1}{3})$ *(Langkah 3),* $R_{32}(-2)$ *and* $R_{31}(-2)$ *(Langkah 4) and* $R_{21}(-1)$ *(Langkah 5) digunakan untuk memperoleh bentuk eselon baris terturun (BEBT) bagi matriks imbuhan:*

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 4 & 8 & 4 \\ 3 & 6 & 5 & 7 & 7 \end{array}\right].$$

Tunjukkan prosedur tersebut secara terperinci untuk memperoleh penyelesaian bagi sistem (guna pembolehubah x, y, z *dan* t*).*

(ii) *Satu sistem persamaan linear mempunyai system setara berikut:*

$$\begin{aligned} x + y \qquad + t &= 4 \\ -2y \qquad - 6t &= -1 \\ z \qquad - t &= 1 \end{aligned}$$

Tunjukkan bahawa terdapat penyelesaian tak terhingga banyak untuk sistem yang berbentuk $(\frac{7}{2}, \frac{1}{2}, 1, 0) + k(2,-3,1,1)$, $k \in \mathbb{R}$

Tunjuk juga bagaimana anda dapat menentusahkannya.

[100 markah]

4. (a) Find the line which best approximates the points $(0,0)$, $(1,2)$, $(2,2)$ using the least squares approximation method.

(b) Let $W = L((1,3,2))$ in $\mathbb{R}^3$.

(i) Determine the basis for $W^\perp$, the orthogonal complement of W.

(ii) Find an orthonormal basis for $W^\perp$.

(iii) Show that the only way for an element v in $\mathbb{R}^3$ can be in both W and $W^\perp$ is when $v = (0,0,0)$.

(c) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation where $\alpha = \{v_1, v_2, v_3\}$ and $\beta = \{u_1, u_2\}$ are ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. It is known that for all $w \in \mathbb{R}^3$, we have $w_\alpha T_{\alpha,\beta} = (w,T)_\beta$.

Use this to find the definition for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with

$\alpha = \{e_1, e_1 + e_2, e_1 + e_2 + e_3\}$, $\beta = \{(1,0),(1,1)\}$ and $T_{\alpha,\beta} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$.

(d) Diagonalize $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$.

[100 marks]

4. (a) *Cari garis yang memberi penghampiran terbaik untuk titik-titik* $(0,0)$, $(1,2)$, $(2,2)$ *dengan menggunakan kaedah penghampiran kuasa dua terkecil.*

(b) *Andai* $W = L((1,3,2))$ *dalam* $\mathbb{R}^3$.

(i) *Tentukan asas untuk* $W^{\perp}$, *pelengkap ortogon untuk* W.

(ii) *Cari asas ortonomal untuk* $W^{\perp}$.

(iii) *Tunjukkan bahawa elemen* v *dalam* $\mathbb{R}^3$ *dapat berada dalam* W *dan juga* $W^{\perp}$ *hanya jika* $v = (0,0,0)$.

(c) *Andai* $T:\mathbb{R}^3 \to \mathbb{R}^2$ *satu transformasi linear dengan* $\alpha = \{v_1, v_2, v_3\}$ *dan* $\beta = \{u_1, u_2\}$ *ialah asas bertertib untuk* $\mathbb{R}^3$ *dan* $\mathbb{R}^2$ *masing-masing. Diketahui bahawa untuk semua* $w \in \mathbb{R}^3$, *kita dapat* $w_\alpha T_{\alpha,\beta} = (w,T)_\beta$.
Guna ini untuk mendapatkan takrif untuk transformasi linear $T:\mathbb{R}^3 \to \mathbb{R}^2$ *dengan* $\alpha = \{e_1, e_1+e_2, e_1+e_2+e_3\}$, $\beta = \{(1,0),(1,1)\}$ *dan* $T_{\alpha,\beta} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$.

(d) *Pepenjurukan* $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$.

[100 markah]

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