UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2008/2009

April/May 2009

MAT 102 – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all ten** [10] questions.

[Arahan: *Jawab* ***semua sepuluh*** *[10] soalan.]*

1. Find the following limits. Use L'Hospital's rule where appropriate.

   (a) $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$ (b) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

   (c) $\lim_{x\to(\pi/2)^-}\sec(5x)\cos(3x)$

   [16 marks]

2. For $n\geq 1$, consider $a_n=\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}$.

   (a) Show that $\{a_n\}$ is decreasing sequence.

   (b) Show that $\frac{1}{2}<\lim_{n\to\infty}a_n<1$.

   (c) Is $\{a_n\}$ bounded? If yes, state its upper and lower bounds. If no, give your reason.

   [18 marks]

3. Test the convergence of the series.

   (a) $\sum_{n=1}^{\infty}\frac{n^n}{(2n)!}$ (Hint: $\lim_{n\to\infty}(1+\frac{1}{n})^n=e^1$) (b) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

   [16 marks]

4. (a) By using a suitable power series representation, find a power series for

   $$f(x)=\frac{1}{(1-x)^2}.$$

   Determine its radius of convergence.

   (b) Use part (a) to find a power series for

   $$f(x)=\frac{1}{(1-x)^3}.$$

   [12 marks]

5. Which of the following integrals is improper? Give your reason. Evaluate it ONLY if it is an improper integral.

   (a) $\int_1^{\infty}\frac{\ln x}{x}dx$ (b) $\int_0^3\frac{1}{x\sqrt{x}}dx$

   [12 marks]

6. Let $f(x,y)=\frac{x-y}{x+y}$, $x+y\neq 0$.

   (a) Show that $\lim_{x\to 0}\lim_{y\to 0}f(x,y)=1$ and $\lim_{y\to 0}\lim_{x\to 0}f(x,y)=-1$.

   (b) What can you say about $\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$?

   [8 marks]

1. *Cari had berikut. Guna petua L'Hospital jika sesuai.*

   (a) $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$ (b) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

   (c) $\lim_{x\to(\pi/2)^-}\sec(5x)\cos(3x)$

   [16 markah]

2. *Bagi* $n\geq 1$, *pertimbangkan* $a_n=\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}$.

   (a) *Tunjukkan bahawa* $\{a_n\}$ *ialah jujukan menyusut.*

   (b) *Tunjukkan bahawa* $\frac{1}{2}<\lim_{n\to\infty}a_n<1$.

   (c) *Adakah* $\{a_n\}$ *terbatas? Jika ya, nyatakan batas atas dan batas bawahnya. Jika tidak, beri alasan anda.*

   [18 markah]

3. *Uji penumpuan siri berikut.*

   (a) $\sum_{n=1}^{\infty}\frac{n^n}{(2n)!}$ *(Petunjuk:* $\lim_{n\to\infty}(1+\frac{1}{n})^n=e^1$*)* (b) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

   [16 markah]

4. (a) *Dengan menggunakan perwakilan siri kuasa yang sesuai, cari siri kuasa untuk*

   $$f(x)=\frac{1}{(1-x)^2}.$$

   Tentukan jejari penumpuannya.

   (b) *Guna bahagian (a) untuk mencari siri kuasa bagi*

   $$f(x)=\frac{1}{(1-x)^3}.$$

   [12 markah]

5. *Adakah kamiran berikut tak wajar? Beri alasan anda. Nilaikannya JIKA kamiran itu kamiran tak wajar.*

   (a) $\int_1^{\infty}\frac{\ln x}{x}dx$ (b) $\int_0^3\frac{1}{x\sqrt{x}}dx$

   [12 markah]

6. *Andaikan* $f(x,y)=\frac{x-y}{x+y}$, $x+y\neq 0$.

   (a) *Tunjukkan bahawa* $\lim_{x\to 0}\lim_{y\to 0}f(x,y)=1$ *dan* $\lim_{y\to 0}\lim_{x\to 0}f(x,y)=-1$.

   (b) *Apa anda boleh katakan tentang* $\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$?

   [8 markah]

7. Consider the function $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

(a) Show that the directional derivative $D_v f(0,0)$ exists in any direction $\vec{v}$.

(b) Find all the first order partial derivative of f.

(c) Find the gradient of f at $(0,0)$.

(d) Is f differentiable at $(0,0)$?

(e) Is f continuous at $(0,0)$?

[22 marks]

8. If $f(x,y)=3x^2+3y^2-4y$, find the extrema of f on $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$.

[8 marks]

9. (a) Evaluate $\iint_R (x^3+4y)\,dxdy$ where R is the region in the xy-plane bounded by $y=x^2$ and $y=2x$.

(b) Reverse the order of the iterated integral $\int_0^4\int_{\sqrt{y}}^2 y\cos(x^5)\,dxdy$.

[20 marks]

10. *True* or *False*

(a) If $\lim_{n\to\infty} a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) If $f:\mathbb{R}^2\to\mathbb{R}$ is continuous at (a,b), then the partial derivatives of f at (a,b) exist.

(c) If f is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)\,dx$ exists.

(d) If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then $\lim_{x\to a} f(x,b)$ and $\lim_{y\to b} f(a,y)$ exist.

(e) If the power series $f(x)=\sum_{n=1}^{\infty} c_n(x-a)^n$ has interval of convergence $[a-R,a+R]$, then f' also has interval of convergence $[a-R,a+R]$.

[10 marks]

7. *Pertimbangkan fungsi* $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

   (a) Tunjukkan bahawa terbitan berarah $D_v f(0,0)$ *wujud di semua arah* $\bar{v}$.

   (b) Cari semua terbitan separa peringkat pertama untuk f.

   (c) Cari gradian untuk f *pada* $(0,0)$.

   (d) Adakah f *terbezakan pada* $(0,0)$*?*

   (e) Adakah f *selanjar pada* $(0,0)$*?*

[22 markah]

8. *Jika* $f(x,y)=3x^2+3y^2-4y$, *cari ekstremum untuk* f *di* $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$.

[8 markah]

9. *(a) Nilaikan* $\iint_R (x^3+4y)\,dxdy$ *dengan* R *ialah rantau yang dibatasi* $y=x^2$ *dan* $y=2x$ *dalam satah xy.*

   (b) Tukar tertib kamiran terlelar $\int_0^4\int_{\sqrt{y}}^2 y\cos(x^5)\,dxdy$.

[20 markah]

10. *Benar atau Salah*

   (a) Jika $\lim_{n\to\infty} a_n$ *wujud, maka* $\sum_{n=1}^{\infty} a_n$ *mencapah.*

   (b) Jika $f:\mathbb{R}^2\to\mathbb{R}$ *selanjar pada* (a,b), *maka terbitan separa untuk* f *pada* (a,b) *wujud.*

   (c) Jika f *selanjar pada* $[a,\infty)$, *maka* $\int_a^{\infty} f(x)\,dx$ *wujud.*

   (d) Jika $\lim_{(x,y)\to(a,b)} f(x,y)$ *wujud, maka* $\lim_{x\to a} f(x,b)$ *dan* $\lim_{y\to b} f(a,y)$ *wujud.*

   (e) Jika siri kuasa $f(x)=\sum_{n=1}^{\infty} c_n(x-a)^n$ *mempunyai selang penumpuan* $[a-R,a+R]$, *maka* f' *juga mempunyai selang penumpuan* $[a-R,a+R]$.

[10 markah]

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