# CONVOLUTION, COEFFICIENT AND RADIUS PROBLEMS OF CERTAIN UNIVALENT FUNCTIONS

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# CONVOLUTION, COEFFICIENT AND RADIUS PROBLEMS OF CERTAIN UNIVALENT FUNCTIONS

by

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IN THE NAME OF ALLAH S.W.T (THE AL-MIGHTY) THE GRACIOUS, THE MOST MERCIFUL.

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# CONTENTS

## SYMBOLS

| Symbol | Description |
|---|---|
| $\mathcal{A}$ | Class of analytic functions of the form |
| | $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U)$ |
| $\arg$ | Argument |
| $\mathcal{C}$ | Complex plane |
| $f * g$ | Convolution or Hadamard product of functions $f$ and $g$ |
| $\mathcal{H}(U)$ | Class of analytic functions in $U$ |
| $\Im$ | Imaginary part of a complex number |
| $\prec$ | Subordinate to |
| $K$ | Class of convex functions in $U$ |
| $K(\alpha)$ | Class of convex functions of order $\alpha$ in $U$ |
| $K(\phi)$ | $\{f \in \mathcal{A}: \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)\}$ |
| $k(z)$ | Koebe function |
| $\mathcal{M}$ | Class of meromorphic functions of the form |
| | $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (z \in U^*)$ |
| $M(\alpha)$ | $\{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad ( -1 \leq B < A \leq 1, \alpha > 1)\}$ |
| $\mathcal{R}$ | Set of all real numbers |
| $\Re$ | Real part of a complex number |
| $R_\alpha$ | Class of prestarlike functions of order $\alpha$ in $U$ |
| $\mathcal{S}$ | Class of all univalent functions in $U$ |
| $S^*$ | Class of starlike functions in $U$ |
| $S^*(\alpha)$ | Class of starlike functions of order $\alpha$ in $U$ |
| $S^*(\phi)$ | $\{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \phi(z)\}$ |
| $\mathcal{S}^*[A, B]$ | $\{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\}$ |
| $SP(\alpha, A, B)$ | $\{f \in \mathcal{A}: e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \cos\alpha\frac{1+Az}{1+Bz} + i\sin\alpha$ |
| | $( -1 \leq B < A \leq 1, 0 \leq \alpha < 1)\}$ |

| | |
|---|---|
| $S_p$ | Class of parabolic starlike functions in $U$ |
| $T$ | Subclass of $\mathcal{A}$ consisting of functions of the form |
| | $f(z) = z - \sum_{n=2}^{\infty} \lvert a_n \rvert z^n \quad (z \in U)$ |
| $U$ | Open unit disk $\{z \in \mathcal{C} : \lvert z \rvert < 1\}$ |
| $U^*$ | Punctured unit disk $U \setminus \{0\}$ |
| $\mathcal{Z}$ | Set of all integers |

## KONVOLUSI, PEKALI DAN MASALAH JEJARI UNTUK FUNGSI UNIVALEN TERTENTU

## ABSTRAK


Suatu fungsi $f$ yang tertakrif dalam cakera unit $U := \{z \in \mathcal{C} : |z| < 1\}$ dalam satah kompleks $\mathcal{C}$ dikatakan univalen jika fungsi tersebut memetakan titik berlainan dalam $U$ ke titik berlainan dalam $\mathcal{C}$. Andaikan $\mathcal{A}$ kelas fungsi analisis ternormalkan yang tertakrif dalam $U$ dan mempunyai siri Taylor dalam bentuk

(0.0.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Suatu fungsi $f$ dikatakan subordinasi kepada suatu fungsi univalen $F$ jika $f(0) = F(0)$ dan $f(U) \subset F(U)$. Hasil darab Hadamard atau konvolusi dua fungsi analisis, $f$ berbentuk yang seperti (0.0.1) dan $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, ditakrif sebagai

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

Andaikan $\mathcal{M}$ kelas fungsi meromorfi, $h$ berbentuk

(0.0.2) $$h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

yang analisis dan univalen dalam $U^* = \{z : 0 < |z| < 1\}$. Konvolusi dua fungsi meromorfi $h$ dan $k$, dengan $h$ diberi dalam bentuk (0.0.2) dan $k(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, ditakrif sebagai

$$(h * k)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n.$$

Dengan menggunakan ciri-ciri konvolusi dan teori subordinasi, beberapa subkelas fungsi meromorfi diperkenalkan. Dengan mensubordinasikan fungsi di dalam kelas ini dengan suatu fungsi cembung ternormalkan yang mempunyai nilai nyata positif, subkelas-subkelas ini merangkumi subkelas klasik meromorfi bak-bintang, cembung, hampir-cembung dan kuasi-cembung. Hubungan kelas dan ciri-ciri konvolusi subkelas-subkelas ini juga dikaji.

Andaikan $T$ subkelas $\mathcal{A}$ yang mengandungi fungsi $t$ dalam bentuk

$$t(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

Silverman [**44**] telah menyiasat subkelas $T$ yang mengandungi fungsi bak-bintang peringkat $\alpha$ dan cembung peringkat $\alpha$ $(0 \leq \alpha < 1)$. Dengan motivasi ini, pelbagai subkelas $T$ telah dikaji. Kelas $T[\{b_k\}_{m+1}^{\infty}, \beta, m]$ yang ditakrifkan secara am akan dikaji dan anggaran pekali, teorem pertumbuhan dan beberapa keputusan untuk kelas ini diperoleh. Keputusan ini merangkumi beberapa keputusan awal sebagai kes khas.

Untuk pemalar kompleks $A$ dan $B$, andaikan $S^*[A, B]$ kelas yang mengandungi fungsi analisis ternormalkan yang mematuhi $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$. Jejari bak-bintang peringkat $\alpha$, jejari bak-bintang kuat dan jejari bak-bintang parabola diperoleh untuk kelas $S^*[A, B]$. Keputusan ini juga merangkumi beberapa keputusan awal.

# CONVOLUTION, COEFFICIENT AND RADIUS PROBLEMS OF CERTAIN UNIVALENT FUNCTIONS

## ABSTRACT

A function $f$ defined on the open unit disc $U := \{z \in \mathcal{C} : |z| < 1\}$ of the complex plane $\mathcal{C}$ is univalent if it maps different points of $U$ to different points in $\mathcal{C}$. Let $\mathcal{A}$ denote the class of analytic functions defined on $U$ which is normalized and has the Taylor series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{0.0.3}$$

The function $f$ is subordinate to a univalent function $F$ if $f(0) = F(0)$ and $f(U) \subset F(U)$. Hadamard product or convolution of two analytic functions $f$, given by (0.0.3) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is given by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let $\mathcal{M}$ denote the class of meromorphic functions $h$ of the form

$$h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{0.0.4}$$

that are analytic and univalent in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. The convolution of two meromorphic functions $h$ and $k$, where $h$ is given by (0.0.4) and $k(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, is given by

$$(h * k)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n.$$

By making use of the properties of convolution and theory of subordination, several subclasses of meromorphic functions are introduced. Subjecting each convoluted-derived function in the class to be subordinated to a given normalized convex function with positive real part, these subclasses extend the classical subclasses of meromorphic starlikeness, convexity, close-to-convexity, and quasi-convexity. Class relations, as well as inclusion and convolution properties of these subclasses are investigated.

Let $T$ denote the subclass of $\mathcal{A}$ consisting of functions $t$ of the form

$$t(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

Silverman [**44**] investigated the subclasses of $T$ consisting of functions which are starlike of order $\alpha$ and convex of order $\alpha$ $(0 \leq \alpha < 1)$. Motivated by his work, many other subclasses of $T$ were studied in the literature. The class $T[\{b_k\}_{m+1}^{\infty}, \beta, m]$ which is defined in a general manner is studied and the coefficient estimate, growth theorem and other results for this class are obtained. Our results contain several earlier results as special cases.

For complex constants A and B, let $S^*[A, B]$ be the class consisting of normalized analytic functions $f$ satisfying $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$. The radius of starlikeness of order $\alpha$, radius of strong-starlikeness, and radius of parabolic-starlikeness are obtained for $S^*[A, B]$. Several known results are shown to be simple consequences of results derived here.

## CHAPTER 1

# INTRODUCTION

The theory of univalent functions is a remarkable area of study. This field which is more often associated with 'geometry' and 'analysis' has raised the interest of many since the beginning of 20th century to recent times. The name univalent functions or schlicht (the German word for simple) functions is given to functions defined on the open unit disc $U := \{z \in \mathcal{C} : |z| < 1\}$ of the complex plane $\mathcal{C}$ that are characterized by the fact that such a function provides one-to-one mapping onto its image.

Let $\mathcal{H}(U)$ be the class of all analytic functions on $U$ and $\mathcal{A}$ denote the class of analytic functions defined on $U$ which is normalized by the condition $f(0) = 0$, $f'(0) = 1$ and has the Taylor series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Geometrically, the function $f$ is univalent if $f(z_1) = f(z_2)$ implies $z_1 = z_2$ in $U$ and is locally univalent at $z_0 \in U$ if it is univalent in some neighborhood of $z_0$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

The Koebe function $k(z) = z/(1-z)^2$ is a univalent function and it plays a very significant role in the study of the class $\mathcal{S}$. In fact, the Koebe function and its rotations $e^{-i\alpha}k(e^{i\alpha}z)$, $\alpha \in \mathcal{R}$ are the only extremal functions for various problems in the class $\mathcal{S}$. For example, the famous findings of Bieberbach. In 1916, Bieberbach proved that if $f \in \mathcal{S}$, then the second coefficient $|a_2| \leq 2$ with equality if and only if $f$ is a rotation of the Koebe function. He also conjectured that $|a_n| \leq n$, $(n = 2, 3, \cdots)$ which is generally valid and this was proved by de Branges **[6]** in 1985.

Several special subclasses of analytic univalent functions play prominent role in the study of this area. Notable among them are the classes of starlike and convex functions.

Let $w_0$ be an interior point of a set $\mathcal{D}$ in the complex plane. The set $\mathcal{D}$ is starlike with respect to $w_0$ if the line segment joining $w_0$ to every other point in $\mathcal{D}$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps $U$ onto a starlike domain, then $f$ is a starlike function. The class of starlike functions with respect to origin is denoted by $S^*$. Analytically,

$$S^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

A set $\mathcal{D}$ in the complex plane is convex if for every pair of points $w_1$ and $w_2$ in the interior of $\mathcal{D}$, the line segment joining $w_1$ and $w_2$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps $U$ onto a convex domain, then $f$ is a convex function. Let $K$ denote the class of all convex functions in $\mathcal{A}$. An analytic description of the class $K$ is given by

$$K := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

The well known connection between these two classes was first observed by Alexander in 1915. The Alexander theorem [**2**] states that for an analytic function $f$, $f(z) \in K$ if and only if $zf'(z) \in S^*$.

Ma and Minda gave a unified presentation of these classes by using the method of subordination. For two functions $f$ and $g$ analytic in $U$, the function $f$ is subordinate to $g$, written

$$f(z) \prec g(z) \quad (z \in U),$$

if there exists a function $w$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function $g$ is univalent in $U$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

With $g(z) = (1+z)/(1-z)$, a function $f \in \mathcal{A}$ is starlike if $zf'(z)/f(z)$ is subordinate to $g$ and is convex if $1+zf''(z)/f'(z)$ is subordinate to $g$. Ma and Minda [**17**] introduced the classes

$$S^*(\phi) = \left\{ f \in \mathcal{A} \,\middle|\, \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

and

$$K(\phi) = \left\{ f \in \mathcal{A} \,\middle|\, 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\},$$

where $\phi$ is an analytic function with positive real part, $\phi(0) = 1$ and $\phi$ maps the unit disk $U$ onto a region starlike with respect to 1.

The convolution or Hadamard product is another interesting exploration of these classes. The convolution of two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is given by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

Polya-Schoenberg [**24**] conjectured that the class of convex functions is preserved under convolution with convex functions. In 1973, Ruscheweyh and Sheil-Small [**36**] proved the Polya-Schoenberg conjecture. In fact, they proved that the classes of starlike functions and convex functions are closed under convolution with convex functions.

Detailed treatment of univalent functions are available in books by Pommerenke [**25**], Duren [**7**] and Goodman [**12**].

## CHAPTER 2

# CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH CONVOLUTION AND DIFFERENTIAL SUBORDINATION

## 2.1. MOTIVATION AND PRELIMINARIES

The convolution or the Hadamard product of two analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is given by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

The geometric series $\sum_{n=1}^{\infty} z^n = z/(1-z)$ acts as the identity element under convolution. The convolution of $f$ with the geometric series $\sum_{n=1}^{\infty} nz^n = z/(1-z)^2$ is given by $\sum_{n=1}^{\infty} na_n z^n$ which is equivalent to $zf'(z)$. In terms of convolution, $f = f * (z/(1-z))$ and $zf' = f * (z/(1-z)^2)$. The well known Alexander's theorem states that a function $f$ is convex if and only if $zf'$ is starlike. Since $zf' = f * (z/(1-z)^2)$, it follows that $f$ is convex if and only if $f * (z/(1-z)^2)$ is starlike. Also, a function $f$ is starlike if $f * (z/(1-z))$ is starlike. These ideas led to the study of the class of all functions $f$ such that $f * g$ is starlike for some fixed function $g$ in $\mathcal{A}$. In this direction, Shanmugam [**41**] introduced and investigated various subclasses of analytic functions by using the convex hull method [**5, 23, 36**] and the method of differential subordination. Ravichandran [**26**] introduced certain classes of analytic functions with respect to $n$-ply symmetric points, conjugate points and symmetric conjugate points and also discussed their convolution properties. Some other related studies were also made in [**3, 22**], and more recently by Shamani *et al.* [**40**].

Let $\mathcal{M}$ denote the class of meromorphic functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{2.1.1}$$

that are analytic and univalent in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. The convolution of two meromorphic functions $f$ and $g$, where $f$ is given by (2.1.1) and

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{2.1.2}$$

is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n.$$

For $0 \leq \alpha < 1$, we recall that the classes of meromorphic starlike, meromorphic convex, meromorphic close-to-convex, meromorphic $\gamma-$convex (Mocanu sense) and meromorphic quasi-convex functions of order $\alpha$, denoted by $\mathcal{M}^s$, $\mathcal{M}^k$, $\mathcal{M}^c$, $\mathcal{M}^k_\gamma$ and $\mathcal{M}^q$ respectively, are defined by

$$\begin{aligned}
\mathcal{M}^s &= \left\{ f \in \mathcal{M} \;\middle|\; -\Re \frac{zf'(z)}{f(z)} > \alpha \right\}, \\
\mathcal{M}^k &= \left\{ f \in \mathcal{M} \;\middle|\; -\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}, \\
\mathcal{M}^c &= \left\{ f \in \mathcal{M} \;\middle|\; -\Re \frac{zf'(z)}{g(z)} > \alpha,\ g(z) \in \mathcal{M}^s \right\}, \\
\mathcal{M}^k_\gamma &= \left\{ f \in \mathcal{M} \;\middle|\; -\Re \left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \alpha \right\}, \\
\mathcal{M}^q &= \left\{ f \in \mathcal{M} \;\middle|\; -\Re \frac{[zf'(z)]'}{g'(z)} > \alpha,\ g(z) \in \mathcal{M}^k \right\}.
\end{aligned} \tag{2.1.3}$$

Motivated by the investigation of Shanmugam [**41**], Ravichandran [**26**], and Ali *et al.* [**3**], several subclasses of meromorphic functions defined by means of convolution with a given fixed meromorphic function are introduced in Section 2.2. These new subclasses extend the classical classes of meromorphic starlike, convex, close-to-convex, $\gamma$-convex, and quasi-convex functions given in (2.1.3). Section 2.3 is devoted to the investigation of the class relations as well as inclusion and convolution properties of these newly defined classes.

We shall need the following definition and results to prove our main results.

Let $S^*(\alpha)$ denote the class of starlike functions of order $\alpha$. The class $R_\alpha$ of prestarlike functions of order $\alpha$ is defined by

$$R_\alpha = \left\{ f \in \mathcal{A} \,\middle|\, f(z) * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha) \right\}$$

for $\alpha < 1$, and

$$R_1 = \left\{ f \in \mathcal{A} \,\middle|\, \Re \frac{f(z)}{z} > \frac{1}{2} \right\}.$$

THEOREM 2.1.1. [**35**, Theorem 2.4] *Let $\alpha \leq 1$, $f \in R_\alpha$ and $g \in S^*(\alpha)$. Then, for any analytic function $H \in \mathcal{H}(U)$,*

$$\frac{f * Hg}{f * g}(U) \subset \overline{co}(H(U))$$

*where $\overline{co}(H(U))$ denotes the closed convex hull of $H(U)$.*

THEOREM 2.1.2. [**8**] *Let $h$ be convex in $U$ and $\beta, \gamma \in \mathcal{C}$ with $\Re(\beta h(z) + \gamma) > 0$. If $p$ is analytic in $U$ with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad \textit{implies} \quad p(z) \prec h(z).$$

We will also be using the following convolution properties.

(i) For two meromorphic functions $f$ and $g$ of the forms $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, we have

$$(f * g)(z) = (g * f)(z)$$

PROOF. For $f$ and $g$ of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, we have

$$\begin{aligned}(f * g)(z) &= \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} b_n a_n z^n \\ &= (g * f)(z). \end{aligned}$$

□

(ii) For two meromorphic functions $f$ and $g$ of the forms $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, we have

$$-z(g * f)'(z) = (g * -zf')(z).$$

PROOF. For $f$ of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$, we have

$$-zf'(z) = \frac{1}{z} - \sum_{n=0}^{\infty} n a_n z^n$$

and hence

$$\begin{aligned}(g * -zf')(z) &= \frac{1}{z} - \sum_{n=0}^{\infty} n a_n b_n z^n \\ &= -z\left(-\frac{1}{z^2} + \sum_{n=0}^{\infty} n a_n b_n z^{n-1}\right) \\ &= -z(g * f)'(z). \end{aligned}$$

□

(iii) For two meromorphic functions $f$ and $g$ of the forms $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, we have

$$z^2(g * f)(z) = (z^2 g * z^2 f)(z).$$

PROOF. For

$$(g * f)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n,$$

we observe that

$$\begin{aligned} z^2(g*f)(z) &= z^2\left(\frac{1}{z}+\sum_{n=0}^{\infty}a_nb_nz^n\right) \\ &= z+z^2\sum_{n=0}^{\infty}a_nb_nz^n \\ &= z^2g(z)*z^2f(z). \end{aligned}$$

□

## 2.2. DEFINITIONS

In this section, various subclasses of $\mathcal{M}$ are defined by means of convolution and subordination. Let $g$ be a fixed function in $\mathcal{M}$, and $h$ be a convex univalent function with positive real part in $U$ and $h(0)=1$.

DEFINITION 2.2.1. The class $\mathcal{M}_g^s(h)$ consists of functions $f\in\mathcal{M}$ satisfying $(g*f)(z)\neq 0$ in $U^*$ and the subordination

$$-\frac{z(g*f)'(z)}{(g*f)(z)}\prec h(z).$$

REMARK 2.2.1. If $g(z)=\frac{1}{z}+\frac{1}{1-z}=\frac{1}{z}+\sum_{n=0}^{\infty}z^n$, then $\mathcal{M}_g^s(h)$ coincides with $\mathcal{M}^s(h)$, where

$$\mathcal{M}^s(h)=\left\{f\in\mathcal{M}\;\middle|\;-\frac{zf'(z)}{f(z)}\prec h(z)\right\}.$$

DEFINITION 2.2.2. The class $\mathcal{M}_g^k(h)$ consists of functions $f\in\mathcal{M}$ satisfying $(g*f)'(z)\neq 0$ in $U^*$ and the subordination

$$-\left\{1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right\}\prec h(z).$$

DEFINITION 2.2.3. The class $\mathcal{M}_g^c(h)$ consists of functions $f\in\mathcal{M}$ such that $(g*\psi)(z)\neq 0$ in $U^*$ for some $\psi\in\mathcal{M}_g^s(h)$ and satisfying the subordination

$$-\frac{z(g*f)'(z)}{(g*\psi)(z)}\prec h(z).$$

Definition 2.2.4. For $\gamma$ real, the class $\mathcal{M}^k_{g,\gamma}(h)$ consists of functions $f \in \mathcal{M}$ satisfying $(g*f)(z) \neq 0$, $(g*f)'(z) \neq 0$ in $U^*$ and the subordination

$$-\left\{\gamma\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right)+(1-\gamma)\left(\frac{z(g*f)'(z)}{(g*f)(z)}\right)\right\} \prec h(z).$$

Remark 2.2.2. For $\gamma = 0$, the class $\mathcal{M}^k_{g,\gamma}(h)$ coincides with the class $\mathcal{M}^s_g(h)$ and for $\gamma = 1$, it reduces to the class $\mathcal{M}^k_g(h)$.

Definition 2.2.5. The class $\mathcal{M}^q_g(h)$ consists of functions $f \in \mathcal{M}$ such that $(g*\varphi)'(z) \neq 0$ in $U^*$ for some $\varphi \in \mathcal{M}^k_g(h)$ and satisfying the subordination

$$\frac{[-z(g*f)'(z)]'}{(g*\varphi)'(z)} \prec h(z).$$

## 2.3. INCLUSION AND CONVOLUTION THEOREM

This section is devoted to the investigation of class relations as well as inclusion and convolution properties of the new subclasses given in Section 2.2.

We will begin with the theorem which is analogue to the well known Alexander's theorem.

Theorem 2.3.1. *The function $f$ is in $\mathcal{M}^k_g(h)$ if and only if $-zf'$ is in $\mathcal{M}^s_g(h)$.*

Proof. Since

$$\begin{aligned}
-\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right) &= -\frac{(g*f)'(z)+z(g*f)''(z)}{(g*f)'(z)} \\
&= -\frac{(z(g*f)'(z))'}{(g*f)'(z)} \\
&= \frac{-z}{-z}\cdot\frac{((g*-zf')(z))'}{(g*f)'(z)} \\
&= -\frac{z(g*-zf')'(z)}{(g*-zf')(z)},
\end{aligned}$$

it follows that $f \in \mathcal{M}^k_g(h)$ if and only if $-zf' \in \mathcal{M}^s_g(h)$. □

Theorem 2.3.2. *Let $h$ be a convex univalent function satisfying $\Re h(z) < 2-\alpha$, $0 \leq \alpha < 1$, and $\phi \in \mathcal{M}$ with $z^2\phi \in R_\alpha$. If $f \in \mathcal{M}^s_g(h)$, then $\phi * f \in \mathcal{M}^s_g(h)$.*

PROOF. Since $f \in \mathcal{M}_g^s(h)$, it follows that

$$-\Re\left\{\frac{z(g*f)'(z)}{(g*f)(z)}\right\} < 2-\alpha$$

or

$$\Re\left\{\frac{z(g*f)'(z)+2(g*f)(z)}{(g*f)(z)}\right\} > \alpha. \tag{2.3.1}$$

The inequality (2.3.1) yields

$$\Re\left\{\frac{z^2z(g*f)'(z)+2z^2(g*f)(z)}{z^2(g*f)(z)}\right\} > \alpha$$

and thus

$$\Re\left\{\frac{z(z^2(g*f))'(z)}{z^2(g*f)(z)}\right\} > \alpha. \tag{2.3.2}$$

Let

$$P(z) = -\frac{z(g*f)'(z)}{(g*f)(z)}.$$

We have

$$\begin{aligned}
-\frac{z(\phi*g*f)'(z)}{(\phi*g*f)(z)} &= \frac{\phi(z)*-z(g*f)'(z)}{\phi(z)*(g*f)(z)}\\
&= \frac{\phi(z)*(g*f)(z)P(z)}{\phi(z)*(g*f)(z)}\cdot\frac{z^2}{z^2}\\
&= \frac{z^2\phi(z)*z^2(g*f)(z)P(z)}{z^2\phi(z)*z^2(g*f)(z)}.
\end{aligned}$$

Inequality (2.3.2) shows that $z^2(g*f) \in S^*(\alpha)$. Therefore Theorem 2.1.1 yields

$$-\frac{z(\phi*g*f)'(z)}{(\phi*g*f)(z)} = \frac{z^2\phi(z)*z^2(g*f)(z)P(z)}{z^2\phi(z)*z^2(g*f)(z)} \in \overline{co}(P(U)),$$

and since $P(z) \prec h(z)$, it follows that

$$-\frac{z(\phi*g*f)'(z)}{(\phi*g*f)(z)} \prec h(z).$$

Hence $\phi * f \in \mathcal{M}_g^s(h)$. □

COROLLARY 2.3.1. $\mathcal{M}_g^s(h) \subset \mathcal{M}_{\phi*g}^s(h)$ *under the conditions of Theorem 2.3.2.*

PROOF. Let $f \in \mathcal{M}_g^s(h)$, then by Theorem 2.3.2 we have $\phi * f \in \mathcal{M}_g^s(h)$ or

$$-\frac{z(\phi * g * f)'(z)}{(\phi * g * f)(z)} \prec h(z),$$

which equivalently yields $f \in \mathcal{M}_{\phi * g}^s(h)$. □

In particular, when $g(z) = \frac{1}{z} + \frac{1}{1-z}$, the following corollary is obtained.

COROLLARY 2.3.2. *Let $h$ and $\phi$ satisfy the conditions of Theorem 2.3.2. If $f \in \mathcal{M}^s(h)$, then $f \in \mathcal{M}_\phi^s(h)$.*

THEOREM 2.3.3. *Let $h$ and $\phi$ satisfy the conditions of Theorem 2.3.2. If $f \in \mathcal{M}_g^k(h)$, then $\phi * f \in \mathcal{M}_g^k(h)$. Equivalently $\mathcal{M}_g^k(h) \subset \mathcal{M}_{\phi * g}^k(h)$.*

PROOF. If $f \in \mathcal{M}_g^k(h)$, then it follows from Theorem 2.3.1 that $-zf' \in \mathcal{M}_g^s(h)$. Theorem 2.3.2 shows that $-z(\phi * f)' = \phi * -zf' \in \mathcal{M}_g^s(h)$. Hence $\phi * f \in \mathcal{M}_g^k(h)$. □

THEOREM 2.3.4. *Let $h$ and $\phi$ satisfy the conditions of Theorem 2.3.2. If $f \in \mathcal{M}_g^c(h)$ with respect to $\psi \in \mathcal{M}_g^s(h)$, then $\phi * f \in \mathcal{M}_g^c(h)$ with respect to $\phi * \psi \in \mathcal{M}_g^s(h)$.*

PROOF. Since $\psi \in \mathcal{M}_g^s(h)$, Theorem 2.3.2 shows that $\phi * \psi \in \mathcal{M}_g^s(h)$ and inequality (2.3.2) yields $z^2(g * \psi) \in S^*(\alpha)$.

Let the function $G$ be defined by

$$G(z) = -\frac{z(g * f)'(z)}{(g * \psi)(z)}.$$

Observe that

$$\begin{aligned}
-\frac{z(\phi * g * f)'(z)}{(\phi * g * \psi)(z)} &= \frac{\phi(z) * -z(g * f)'(z)}{\phi(z) * (g * \psi)(z)} \\
&= \frac{\phi(z) * (g * \psi)(z)G(z)}{\phi(z) * (g * \psi)(z)} \cdot \frac{z^2}{z^2} \\
&= \frac{z^2\phi(z) * z^2(g * \psi)(z)G(z)}{z^2\phi(z) * z^2(g * \psi)(z)}.
\end{aligned}$$

Since $z^2\phi \in R_\alpha$ and $z^2(g * \psi) \in S^*(\alpha)$, it follows from Theorem 2.1.1 that

$$-\frac{z(\phi * g * f)'(z)}{(\phi * g * \psi)(z)} = \frac{z^2\phi(z) * z^2(g * \psi)(z)G(z)}{z^2\phi(z) * z^2(g * \psi)(z)} \prec h(z). \tag{2.3.3}$$

Thus $\phi * f \in \mathcal{M}_g^c(h)$ with respect to $\phi * \psi \in \mathcal{M}_g^s(h)$. □

COROLLARY 2.3.3. $\mathcal{M}_g^c(h) \subset \mathcal{M}_{\phi * g}^c(h)$ *under the conditions of Theorem 2.3.2.*

PROOF. If $f \in \mathcal{M}_g^c(h)$ with respect to $\psi \in \mathcal{M}_g^s(h)$, then Theorem 2.3.4 shows that $\phi * f \in \mathcal{M}_g^c(h)$ with respect to $\phi * \psi \in \mathcal{M}_g^s(h)$ which is equivalent to

$$-\frac{z(\phi * g * f)'(z)}{(\phi * g * \psi)(z)} \prec h(z)$$

or $f \in \mathcal{M}_{\phi * g}^c(h)$. Hence $\mathcal{M}_g^c(h) \subset \mathcal{M}_{\phi * g}^c(h)$. □

THEOREM 2.3.5. *Let* $\Re(\gamma h(z)) < 0$. *Then*

(i) $\mathcal{M}_{g,\gamma}^k(h) \subset \mathcal{M}_g^s(h)$,
(ii) $\mathcal{M}_{g,\gamma}^k(h) \subset \mathcal{M}_{g,\beta}^k(h)$ *for* $\gamma < \beta \leq 0$.

PROOF. Define the function $J_g(\gamma; f)$ by

$$J_g(\gamma; f)(z) = -\left\{\gamma\left(1 + \frac{z(g * f)''(z)}{(g * f)'(z)}\right) + (1-\gamma)\left(\frac{z(g * f)'(z)}{(g * f)(z)}\right)\right\}.$$

For $f \in \mathcal{M}_{g,\gamma}^k(h)$, it follows that $J_g(\gamma; f)(z) \prec h(z)$. Let the function $P$ be defined by

$$P(z) = -\frac{z(g * f)'(z)}{(g * f)(z)}. \tag{2.3.4}$$

The logarithmic derivative of $P(z)$ yields

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \frac{(g * f)''(z)}{(g * f)'(z)} - \frac{(g * f)'(z)}{(g * f)(z)}, \tag{2.3.5}$$

and multiplication with $-\gamma z$ to (2.3.5) gives

$$-\gamma\frac{zP'(z)}{P(z)} = -\gamma - \gamma\frac{z(g * f)''(z)}{(g * f)'(z)} + \gamma\frac{z(g * f)'(z)}{(g * f)(z)}. \tag{2.3.6}$$

Adding $P(z)$ to (2.3.6) yields

$$
\begin{aligned}
P(z) - \gamma \frac{zP'(z)}{P(z)} &= -\gamma - \gamma \frac{z(g*f)''(z)}{(g*f)'(z)} + \gamma \frac{z(g*f)'(z)}{(g*f)(z)} - \frac{z(g*f)'(z)}{(g*f)(z)} \\
&= -\left\{ \gamma \left( 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} \right) + (1-\gamma) \left( \frac{z(g*f)'(z)}{(g*f)(z)} \right) \right\} \\
&= J_g(\gamma; f)(z) \qquad (2.3.7)
\end{aligned}
$$

and hence

$$P(z) - \gamma \frac{zP'(z)}{P(z)} \prec h(z).$$

(i) Since $\Re(\gamma h(z)) < 0$ and

$$P(z) - \gamma \frac{zP'(z)}{P(z)} \prec h(z),$$

Theorem 2.1.2 yields $P(z) \prec h(z)$. Hence $f \in \mathcal{M}_g^s(h)$ and this concludes that $\mathcal{M}_{g,\gamma}^k(h) \subset \mathcal{M}_g^s(h)$.

(ii) The logarithmic derivative of $P(z)$ and multiplication of $z$ yields

$$\frac{zP'(z)}{P(z)} = 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} + P(z). \qquad (2.3.8)$$

From (2.3.7), it follows that

$$\frac{zP'(z)}{P(z)} = \frac{P(z) - J_g(\gamma; f)}{\gamma}. \qquad (2.3.9)$$

Let

$$J_g(\beta; f)(z) = -\left\{ \beta \left( 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} \right) + (1-\beta) \left( \frac{z(g*f)'(z)}{(g*f)(z)} \right) \right\}. \qquad (2.3.10)$$

Substituting (2.3.8) and (2.3.9) in (2.3.10) yield

$$J_g(\beta; f)(z) = \left( 1 - \frac{\beta}{\gamma} \right) P(z) + \frac{\beta}{\gamma} J_g(\gamma; f)(z). \qquad (2.3.11)$$

We know that $J_g(\gamma; f)(z) \prec h(z)$ and $P(z) \prec h(z)$ from (i) and since $0 < \frac{\beta}{\gamma} < 1$ and $h(U)$ is convex, we deduce that $J_g(\beta; f)(z) \in h(U)$. Therefore, $J_g(\beta; f)(z) \prec h(z)$ and hence $\mathcal{M}_{g,\gamma}^k(h) \subset \mathcal{M}_{g,\beta}^k(h)$ for $\gamma < \beta \leq 0$ □

Corollary 2.3.4. *The class $\mathcal{M}_g^k(h)$ is a subset of the class $\mathcal{M}_g^q(h)$.*

PROOF. Let $f \in \mathcal{M}_g^k(h)$ and by taking $f = \varphi$, it follows from the definition of the class $\mathcal{M}_g^q(h)$ that $\mathcal{M}_g^k(h) \subset \mathcal{M}_g^q(h)$ . □

THEOREM 2.3.6. *The function $f$ is in $\mathcal{M}_g^q(h)$ if and only if $-zf'$ is in $\mathcal{M}_g^c(h)$.*

PROOF. If $f \in \mathcal{M}_g^q(h)$, then there exists $\varphi \in \mathcal{M}_g^k(h)$ such that

$$\frac{[-z(g*f)'(z)]'}{(g*\varphi)'(z)} \prec h(z).$$

Note that

$$\frac{[-z(g*f)'(z)]'}{(g*\varphi)'(z)} = \frac{[(g*-zf')(z)]'}{(g*\varphi)'(z)} \cdot \frac{-z}{-z} = \frac{-z(g*-zf')'(z)}{(g*-z\varphi')(z)}.$$

Hence

$$\frac{-z(g*-zf')'(z)}{(g*-z\varphi')(z)} \prec h(z).$$

Since $\varphi \in \mathcal{M}_g^k(h)$, by Theorem 2.3.1, $-z\varphi' \in \mathcal{M}_g^s(h)$. Thus by definition 2.2.3, we have $-zf' \in \mathcal{M}_g^c(h)$.

Conversely, if $-zf' \in \mathcal{M}_g^c(h)$, then

$$-\frac{z(g*-zf')'(z)}{(g*\varphi_1)(z)} \prec h(z)$$

for some $\varphi_1 \in \mathcal{M}_g^s(h)$. Let $\varphi \in \mathcal{M}_g^k(h)$ be such that $-z\varphi' = \varphi_1 \in \mathcal{M}_g^s(h)$. The proof is completed by observing that

$$\frac{[-z(g*f)'(z)]'}{(g*\varphi)'(z)} = -\frac{z(g*-zf')'(z)}{(g*-z\varphi')(z)} \prec h(z). \qquad \square$$

COROLLARY 2.3.5. *Let $h$ and $\phi$ satisfy the conditions of Theorem 2.3.2. If $f \in \mathcal{M}_g^q(h)$, then $\phi * f \in \mathcal{M}_g^q(h)$.*

PROOF. If $f \in \mathcal{M}_g^q(h)$, Theorem 2.3.6 gives $-zf' \in \mathcal{M}_g^c(h)$. Theorem 2.3.4 next gives $\phi*-zf' = -z(\phi*f)' \in \mathcal{M}_g^c(h)$. Thus, Theorem 2.3.6 yields $\phi*f \in \mathcal{M}_g^q(h)$. □

COROLLARY 2.3.6. $\mathcal{M}_g^q(h) \subset \mathcal{M}_{\phi*g}^q(h)$ *under the conditions of Theorem 2.3.2.*

PROOF. If $f \in \mathcal{M}_g^q(h)$, it follows from Corollary 2.3.5 that $\phi * f \in \mathcal{M}_g^q(h)$. The subordination

$$\frac{[-z(\phi * g * f)'(z)]'}{(\phi * g * \varphi)'(z)} \prec h(z)$$

gives $f \in \mathcal{M}_{\phi * g}^q(h)$. Therefore $\mathcal{M}_g^q(h) \subset \mathcal{M}_{\phi * g}^q(h)$. $\square$

# CHAPTER 3

# A GENERALIZED CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

## 3.1. MOTIVATION AND PRELIMINARIES

Let $T$ denote the subclass of $\mathcal{A}$ consisting of functions $f$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

A function $f \in T$ is called a function with negative coefficients. In [**44**], Silverman investigated the subclasses of $T$ which were denoted by $TS^*(\alpha)$ and $TK(\alpha)$ respectively consisting of functions which are starlike of order $\alpha$ and convex of order $\alpha$ $(0 \leq \alpha < 1)$. He proved the following:

THEOREM 3.1.1. [**44**] *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *Then* $f \in TS^*(\alpha)$ *if and only if* $\sum_{n=2}^{\infty}(n-\alpha)|a_k| \leq 1-\alpha$.

COROLLARY 3.1.1. [**44**] *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *Then* $f \in TK(\alpha)$ *if and only if* $\sum_{n=2}^{\infty} n(n-\alpha)|a_k| \leq 1-\alpha$.

The work of Silverman has brought special interest in the exploration of the functions with negative coefficients. Motivated by his work, many other subclasses of $T$ were studied in the literature. For example, the class $\mathcal{U}(k,\tau,\alpha)$ defined below, was studied by Shanmugam [**43**].

DEFINITION 3.1.1. [**43**] For $0 \leq \tau \leq 1$, $0 \leq \alpha < 1$ and $k \geq 0$, let $\mathcal{U}(k,\tau,\alpha)$, consist of functions $f \in T$ satisfying the condition

$$\Re\left\{\frac{\tau z^3 f'''(z) + (1+2\tau)z^2 f''(z) + z f'(z)}{\tau z^2 f''(z) + z f'(z)}\right\}$$

$$> k\left|\frac{\tau z^3 f'''(z)+(1+2\tau)z^2 f''(z)+zf'(z)}{\tau z^2 f''(z)+zf'(z)}-1\right|+\alpha.$$

The class $\mathcal{U}(k,\tau,\alpha)$ contains well known classes as special cases. In particular, $\mathcal{U}(k,0,0)$ is the class of $k-$uniformly convex function introduced and studied by Kanas and Wisniowska [**15**] and $\mathcal{U}(0,0,\alpha)$ coincides with the class $TK(\alpha)$ studied in [**44**]. In [**43**], Shanmugam proved the coefficients bounds, extreme points as well as radius of starlikeness and convexity theorem for functions in $\mathcal{U}(k,\tau,\alpha)$.

Kadioğlu in [**13**] extended the results by Silverman by defining the class $T_s(\alpha)$ with the use of Sălăgean derivative operator and proved some properties of the functions in this class. The Sălăgean derivatives operator was introduced in [**38**], where for $f(z)\in\mathcal{A}$,

$$D^0 f(z)=f(z),\quad D^1 f(z)=zf'(z)$$

and

$$D^s f(z)=D(D^{s-1}f(z))\quad (s=1,2,3,\ldots).$$

Observe that

$$D^0 f(z)=f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,$$

$$D^1 f(z)=zf'(z)=z\left(1-\sum_{n=2}^{\infty}n|a_n|z^{n-1}\right)=z-\sum_{n=2}^{\infty}n|a_n|z^n$$

and

$$D^2 f(z)=D(D^1 f(z))=z(zf'(z))'=z-\sum_{n=2}^{\infty}n^2|a_n|z^n$$

$$\vdots$$

Hence

$$D^s f(z)=D(D^{s-1}f(z))=z-\sum_{n=2}^{\infty}n^s|a_n|z^n.$$

Kadioğlu defined the class $T_s(\alpha)$ as the following:

Definition 3.1.2. [**13**]

$$T_s(\alpha)=\left\{f\in T:\Re\left\{\frac{D^{s+1}f(z)}{D^s f(z)}\right\}>\alpha\right\}$$

Note that $T_0(\alpha) = TS^*(\alpha)$ and $T_1(\alpha) = TK(\alpha)$. He proved the following:

THEOREM 3.1.2. **[13]** *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ *is in* $T_s(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n^{s+1} - n^s\alpha)|a_n| \leq 1 - \alpha.$$

Ahuja **[1]** defined the class $T_\lambda(\alpha)$ with the use of Ruscheweyh derivative operator. The Ruscheweyh derivative operator $D^\lambda$ is defined using the Hadamard product or convolution by

$$D^\lambda f = \frac{z}{(1-z)^{\lambda+1}} * f \quad \text{for } \lambda \geq -1.$$

DEFINITION 3.1.3. **[1]** A function $f \in T$ is said to be in the class $T_\lambda(\alpha)$ if it satisfies

$$\Re\left\{z\frac{(D^\lambda f(z))'}{D^\lambda f(z)}\right\} > \alpha$$

for $\lambda > -1$ and $\alpha < 1$.

By letting $\lambda = 0$ and $\lambda = 1$, the class $T_\lambda(\alpha)$ will reduce to the class $TS^*(\alpha)$ and $TK(\alpha)$ respectively. The following theorem was proved in **[1]**.

THEOREM 3.1.3. *A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *is in* $T_\lambda(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)B_n(\lambda)a_n \leq 1 - \alpha$$

*where*

$$B_n(\lambda) = \frac{(\lambda+1)_{n-1}}{(n-1)!} = \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+n-1)}{(n-1)!}.$$

The above defined classes as well as numerous other classes of functions can be investigated in a unified manner. For this purpose, we study the following class of $T[\{b_k\}_{m+1}^{\infty}, \beta, m]$.

DEFINITION 3.1.4. Let $b_{m+1} > 0$, $b_{m+1} \le b_k$ for $k \ge m+1$. Also let $\beta \ge 0$ and $m \ge 1$ be an integer. The class $T[\{b_k\}_{m+1}^{\infty}, \beta, m]$ is defined by

$$T[\{b_k\}_{m+1}^{\infty}, \beta, m] = \left\{ f = z - \sum_{k=m+1}^{\infty} a_k z^k \;\middle|\; a_k \ge 0 \text{ for } k \ge m+1, \sum_{k=m+1}^{\infty} b_k a_k \le \beta \right\}.$$

For convenience, we will denote $T[\{b_k\}_{m+1}^{\infty}, \beta, m]$ as $T[b_k, \beta, m]$ and adopt this notation hereafter.

For $m = 1$, the class $T[b_k, \beta, m]$ coincides with the class introduced by Frasin [**9**]. In his paper, Frasin investigated the partial sums of functions belonging to this class. There are many subclasses of $T$ studied by various authors of which several can be represented as $T[b_k, \beta, m]$ with suitable choices of $b_k, \beta$ and $m$ (see example below) (also see [**4, 14, 18, 19, 20, 34, 37, 39**]).

EXAMPLE 3.1.1.

(1) $T[\frac{k-\alpha}{1-\alpha}\frac{(\lambda+1)_{k-1}}{(k-1)!}(\lambda), 1, 1] = T_{\lambda}(\alpha)$, [**1**]

(2) $T[n^{s+1} - \alpha n^s, 1-\alpha, 1] = T_s(\alpha)$, [**13**]

(3) $T[n((n-\alpha)(\tau n - \tau + 1) + k(\tau + n - 1)), 1-\alpha, 1] = U(k, \tau, \alpha)$, [**43**]

(4) $T[n-\alpha, 1-\alpha, 1] = TS^*(\alpha)$, [**44**]

(5) $T[n(n-\alpha), 1-\alpha, 1] = TK(\alpha)$. [**44**]

We will be using Example 3.1.1 later to prove the corollaries in this chapter.

In this chapter we obtain the coefficient estimates, growth theorem, distortion theorem, covering theorem and closure theorem for the class $T[b_k, \beta, m]$. We also consider the extreme points and investigate radius problem for this class.

## 3.2. COEFFICIENT ESTIMATE

We begin with the theorem which gives us the estimate for the coefficient of functions in the class $T[b_k, \beta, m]$.

THEOREM 3.2.1. *If $f \in T[b_k, \beta, m]$, then*

$$a_j \leq \frac{\beta}{b_j}, \quad (j = m+1, m+2, \ldots)$$

*with the equality only for functions of the form $f_j(z) = z - \frac{\beta}{b_j} z^j$.*

PROOF. Let $f \in T[b_k, \beta, m]$, then by definition, we have

$$\sum_{k=m+1}^{\infty} b_k a_k \leq \beta.$$

Hence, it follows that

$$b_j a_j \leq \sum_{k=m+1}^{\infty} b_k a_k \leq \beta \quad (j = m+1, m+2, \ldots)$$

or

$$a_j \leq \frac{\beta}{b_j}.$$

It is clear that for the function of the form

$$f_j(z) = z - \frac{\beta}{b_j} z^j \in T[b_k, \beta, m]$$

we have

$$a_j = \frac{\beta}{b_j}.$$

□

COROLLARY 3.2.1. **[44]** *If $f \in TS^*(\alpha)$, then*

$$a_n \leq \frac{1-\alpha}{n-\alpha},$$

*with equality for function of the form*

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha} z^n.$$

COROLLARY 3.2.2. **[43]** *If $f \in \mathcal{U}(k, \tau, \alpha)$, then*

$$a_n \leq \frac{1-\alpha}{n[(n-\alpha)(\tau n - \tau + 1) + k(\tau + n - 1)]},$$

*with equality for the function of the form*

$$f(z) = z - \frac{1-\alpha}{n[(n-\alpha)(\tau n - \tau + 1) + k(\tau + n - 1)]} z^n.$$

COROLLARY 3.2.3. [1] *If $f \in T_\lambda(\alpha)$, then*

$$a_k \leq \frac{1-\alpha}{(k-\alpha)B_k(\lambda)},$$

*with equality for function of the form*

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)B_k(\lambda)} z^k.$$

## 3.3. GROWTH THEOREM

We now prove the growth theorem for the functions in the class $T[b_k, \beta, m]$.

THEOREM 3.3.1. *If $f \in T[b_k, \beta, m]$, then*

$$r - \frac{\beta}{b_{m+1}} r^{m+1} \leq |f(z)| \leq r + \frac{\beta}{b_{m+1}} r^{m+1}, \quad |z| = r < 1.$$

*with equality for*

$$f(z) = z - \frac{\beta}{b_{m+1}} z^{m+1} \tag{3.3.1}$$

*at $z = r$ for the lower bound and $z = re^{\frac{i\pi(2p+1)}{m}}$, $(p \in \mathcal{Z}^+)$ for the upper bound.*

PROOF. For $f \in T[b_k, \beta, m]$, we have

$$\sum_{k=m+1}^{\infty} b_k a_k \leq \beta$$

and since $b_{m+1} \leq b_k$ for $k \geq m+1$, we have

$$b_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=m+1}^{\infty} b_k a_k \leq \beta$$

or

$$\sum_{k=m+1}^{\infty} a_k \leq \frac{\beta}{b_{m+1}}. \tag{3.3.2}$$

Let $|z| = r$. Since $f = z - \sum_{k=m+1}^{\infty} a_k z^k \in T[b_k, \beta, m]$, we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{k=m+1}^{\infty} a_k r^k \\ &\leq r + r^{m+1} \sum_{k=m+1}^{\infty} a_k. \end{aligned}$$

By using (3.3.2), we obtain

$$|f(z)| \leq r + \frac{\beta}{b_{m+1}} r^{m+1}$$

and similarly

$$|f(z)| \geq r - \frac{\beta}{b_{m+1}} r^{m+1}. \qquad \square$$

Corollary 3.3.1. **[43]** *If $f \in \mathcal{U}(k, \tau, \alpha)$, then*

$$r - \frac{1-\alpha}{2(1+\tau)(2+k-\alpha)} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2(1+\tau)(2+k-\alpha)} r^2 \quad (|z| = r)$$

*with equality for*

$$f(z) = z - \frac{1-\alpha}{2(1+\tau)(2+k-\alpha)} z^2.$$

Corollary 3.3.2. **[13]** *If $f \in T_s(\alpha)$, then*

$$r - \frac{1-\alpha}{2^{s+1} - 2^s \alpha} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2^{s+1} - 2^s \alpha} r^2 \quad (|z| = r)$$

*with equality for*

$$f(z) = z - \frac{1-\alpha}{2^{s+1} - 2^s \alpha} z^2.$$

Corollary 3.3.3. **[1]** *If $f \in T_\lambda(\alpha)$, then*

$$r - \frac{1-\alpha}{(2-\alpha)(\lambda+1)} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(2-\alpha)(\lambda+1)} r^2 \quad (|z| = r)$$

*with equality for*

$$f(z) = z - \frac{1-\alpha}{(2-\alpha)(\lambda+1)} z^2.$$

## 3.4. COVERING THEOREM

THEOREM 3.4.1. *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{\beta}{b_{m+1}}$$

*by any* $f \in T[b_k, \beta, m]$. *The result is sharp for the function* $f$ *given in* (3.3.1).

PROOF. The proof follows by letting $r \to 1$ in Theorem 3.3.1. □

COROLLARY 3.4.1. **[44]** *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{1}{2 - \alpha}$$

*by any* $f \in TS^*(\alpha)$. *The theorem is sharp with extremal function* $f(z) = z - \frac{1-\alpha}{2-\alpha} z^2$.

COROLLARY 3.4.2. **[44]** *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{3 - \alpha}{2(2 - \alpha)}$$

*by any* $f \in TK(\alpha)$. *The theorem is sharp with extremal function* $f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2$.

COROLLARY 3.4.3. **[13]** *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{1 - \alpha}{2^{s+1} - 2^s \alpha}$$

*by any* $f \in T_s(\alpha)$. *The theorem is sharp with extremal function* $f(z) = z - \frac{1-\alpha}{2^{s+1}-2^s\alpha} z^2$.

COROLLARY 3.4.4. **[1]** *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{1 - \alpha}{(2 - \alpha)(\lambda + 1)}$$

*by any* $f \in T_\lambda(\alpha)$. *The theorem is sharp with extremal function* $f(z) = z - \frac{1-\alpha}{(2-\alpha)(\lambda+1)} z^2$.

## 3.5. DISTORTION THEOREM

The distortion theorem for the functions in the class $T[b_k, \beta, m]$ is given in the following theorem.

THEOREM 3.5.1. *If* $f \in T[b_k, \beta, m]$*, then*

$$1 - \frac{\beta(m+1)}{b_{m+1}} r^m \leq |f'(z)| \leq 1 + \frac{\beta(m+1)}{b_{m+1}} r^m, \quad |z| = r < 1$$

*where* $\frac{b_{m+1}}{m+1} \leq \frac{b_k}{k}$*. The result is sharp for the function* $f$ *given in* (3.3.1).

PROOF. For $f \in T[b_k, \beta, m]$, we have

$$\sum_{k=m+1}^{\infty} \frac{b_k}{k} k a_k = \sum_{k=m+1}^{\infty} b_k a_k \leq \beta.$$

Since $\frac{b_{m+1}}{m+1} \leq \frac{b_k}{k}$ for $k \geq m+1$, we obtain

$$\sum_{k=m+1}^{\infty} \frac{b_{m+1}}{m+1} k a_k \leq \sum_{k=m+1}^{\infty} \frac{b_k}{k} k a_k \leq \beta \tag{3.5.1}$$

or equivalently

$$\frac{b_{m+1}}{m+1} \sum_{k=m+1}^{\infty} k a_k \leq \sum_{k=m+1}^{\infty} b_k a_k \leq \beta.$$

Thus we have

$$\sum_{k=m+1}^{\infty} k a_k \leq \frac{\beta(m+1)}{b_{m+1}}. \tag{3.5.1}$$

Since $f(z) = z - \sum_{k=m+1}^{\infty} a_k z^k$, we have

$$f'(z) = 1 - \sum_{k=m+1}^{\infty} k a_k z^{k-1}.$$

Let $|z| = r$, then

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{k=m+1}^{\infty} k a_k r^{k-1} \\ &\leq 1 + r^m \sum_{k=m+1}^{\infty} k a_k \\ &\leq 1 + \frac{\beta(m+1)}{b_{m+1}} r^m \end{aligned}$$