CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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CONTENTS

SYMBOLS

Symbol	Description
$\mathcal{A}_p$	Class of all p-valent analytic functions of the form $f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \quad (z \in U)$
$\mathcal{A} := \mathcal{A}_1$	Class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U)$
$\mathcal{S}$	Class of all normalized univalent functions of the form $f(z) = z + a_2 z^2 + \cdots \quad z \in U$
U	Open unit disk $\{z \in \mathcal{C} : \|z\| < 1\}$
U^*	Punctured unit disk $U \setminus \{0\}$
$U(\mathfrak{p})$	Class of meromorphic functions $f(z)$ in unit disk U with a simple pole at $z = \mathfrak{p}$, $\mathfrak{p} > 0$
$\mathcal{P}$	Class of functions $P(z)$ which are meromorphic in U
$K(\mathfrak{p})$	Class of functions which belong to $U(\mathfrak{p})$ and map $\|z\| < r < \rho$ (for some $\mathfrak{p} < \rho < 1$) onto the complement of a convex set
$\mathcal{H}(U)$	Class of analytic functions in U
$\mathcal{C}$	Complex plane
K	Class of convex functions in U
$K(\alpha)$	Class of convex functions of order α in U
$\mathcal{S}^*$	Class of starlike functions in U
$S^*(\alpha)$	Class of starlike functions of order α in U
C	Class of close-to-convex functions in U
$\mathcal{UCV}$	Class of uniformly convex functions in U
$\mathcal{S}_p$	Class of parabolic starlike functions in U
$f * g$	Convolution or Hadamard product of functions f and g
$\prec$	Subordinate to

$k(z)$	Koebe function
$\mathcal{N}$	Set of all positive integers
$\mathcal{R}$	Set of all real numbers
$\Re$	Real part of a complex number
$\Im$	Imaginary part of a complex number
$\mathcal{Z}$	Set of all integers

KONVOLUSI DAN MASALAH PEKALI BAGI FUNGSI MULTIVALEN DITAKRIF DENGAN SUBORDINASI

ABSTRAK

Andaikan $\mathcal{C}$ satah kompleks, $U = \{z \in \mathcal{C} : |z| < 1\}$ cakera unit terbuka dalam $\mathcal{C}$ dan $\mathcal{H}(U)$ kelas fungsi analisis dalam U. Andaikan juga $\mathcal{A}$ kelas fungsi analisis f dalam U yang ternormalkan dengan $f(0) = 0$ dan $f'(0) = 1$. Fungsi $f \in \mathcal{A}$ mempunyai siri Taylor berbentuk

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).$$

Andaikan $\mathcal{A}_p$ $(p \in \mathcal{N})$ kelas fungsi analisis f berbentuk

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in U)$$

dengan $\mathcal{A} := \mathcal{A}_1$.

Pertimbangkan dua fungsi

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \quad \text{dan} \quad g(z) = z^p + b_{p+1} z^{p+1} + \cdots$$

dalam $\mathcal{A}_p$. Hasil darab Hadamard (atau konvolusi) untuk f dan g ialah fungsi $f * g$ berbentuk

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

Dalam Bab 1, kelas-kelas teritlak bak-bintang multivalen, cembung, hampir-cembung dan kuasi-cembung diperkenalkan. Kelas-kelas tersebut memberi kaedah penyatuan untuk pelbagai subkelas yang diketahui sebelum ini. Ciri-ciri konvolusi dan inklusi diterbitkan dengan menggunakan kaedah hul cembung dan subordinasi pembeza.

Dalam Bab 2, batas untuk pekali fungsian Fekete-Szegö bersekutu dengan transformasi punca ke-k $[f(z^k)]^{1/k}$ fungsi-fungsi analisis ternormalkan f tertakrif dalam U

diperoleh untuk kelas-kelas fungsi berikut:

$$\begin{aligned} R_b(\varphi) &:= \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\}, \\ S^*(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}, \\ L(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\}, \\ M(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}, \end{aligned}$$

dengan $b \in \mathcal{C} \setminus \{0\}$ dan $\alpha \geq 0$. Masalah yang serupa dikaji untuk fungsi $z/f(z)$ bagi f di dalam kelas-kelas fungsi tertentu.

Dalam Bab 3, beberapa subkelas fungsi univalen meromorfi dalam U diitlakkan. Andaikan $U(\mathfrak{p})$ kelas fungsi-fungsi univalen meromorfi ternormalkan f dalam U dengan kutub ringkas pada $z = \mathfrak{p}$, $\mathfrak{p} > 0$. Andaikan ϕ suatu fungsi dengan bahagian nyata positif dalam U, $\phi(0) = 1$, $\phi'(0) > 0$, yang memetakan U keseluruh rantau bak-bintang terhadap 1 dan simetri terhadap paksi nyata. Kelas $\sum^*(\mathfrak{p}, w_0, \phi)$ mengandungi fungsi $f \in U(\mathfrak{p})$, meromorfi bak-bintang terhadap w_0 sedemikian hingga

$$-\left(\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

Kelas $\sum(\mathfrak{p}, \phi)$ mengandungi fungsi $f \in U(\mathfrak{p})$, meromorfi cembung sedemikian hingga

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

Batas untuk w_0 dan beberapa pekali untuk f di dalam $\sum^*(\mathfrak{p}, w_0, \phi)$ dan $\sum(\mathfrak{p}, \phi)$ diperoleh.

CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

ABSTRACT

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$ and $\mathcal{H}(U)$ be the class of analytic functions defined in U. Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $U := \{z \in \mathcal{C} : |z| < 1\}$, and normalized by $f(0) = 0$, and $f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

Let $\mathcal{A}_p$ $(p \in \mathcal{N})$ be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

with $\mathcal{A} := \mathcal{A}_1$.

Consider two functions

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \quad \text{and} \quad g(z) = z^p + b_{p+1} z^{p+1} + \cdots$$

in $\mathcal{A}_p$. The Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

In Chapter 1, the general classes of multi-valent starlike, convex, close-to-convex and quasi-convex functions are introduced. These classes provide a unified treatment to various known subclasses. Inclusion and convolution properties are derived using the methods of convex hull and differential subordination.

In Chapter 2, bounds for the Fekete-Szegö coefficient functional associated with the k-th root transform $[f(z^k)]^{1/k}$ of normalized analytic functions f defined on U

are derived for the following classes of functions:

$$R_b(\varphi) := \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\},$$
$$S^*(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\},$$
$$L(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$
$$M(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

where $b \in \mathcal{C} \setminus \{0\}$ and $\alpha \geq 0$. A similar problem is investigated for functions $z/f(z)$ when f belongs to a certain class of functions.

In Chapter 3, some subclasses of meromorphic univalent functions in the unit disk U are extended. Let $U(\mathfrak{p})$ denote the class of normalized univalent meromorphic functions f in U with a simple pole at $z = \mathfrak{p}$, $\mathfrak{p} > 0$. Let ϕ be a function with positive real part on U, $\phi(0) = 1$, $\phi'(0) > 0$, which maps U onto a region starlike with respect to 1 and which is symmetric with respect to the real axis. The class $\sum^*(\mathfrak{p}, w_0, \phi)$ consists of functions $f \in U(\mathfrak{p})$ meromorphic starlike with respect to w_0 and satisfying

$$-\left(\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

The class $\sum(\mathfrak{p}, \phi)$ consists of functions $f \in U(\mathfrak{p})$ meromorphic convex and satisfying

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

The bounds for w_0 and some initial coefficients of f in $\sum^*(\mathfrak{p}, w_0, \phi)$ and $\sum(\mathfrak{p}, \phi)$ are obtained.

CHAPTER 1

INTRODUCTION

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$ and $\mathcal{H}(U)$ be the class of analytic functions defined on U. Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $U := \{z \in \mathcal{C} : |z| < 1\}$, and normalized by $f(0) = 0$, and $f'(0) = 1$.

A function f is said to be univalent in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

The Koebe function $k(z) = z/(1-z)^2$ maps U onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$ and is univalent. It plays a very important role in the study of the class $\mathcal{S}$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, for $\beta \in \mathcal{R}$, are the extremal functions for various problems in the class $\mathcal{S}$. For example, the de Branges Theorem tells that if $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is analytic and univalent in U, the coefficients satisfy $|a_n| \leq n$, $(n = 2, 3, \ldots)$ with equality if and only if f is a rotation of the Koebe function. This theorem was conjectured by Bieberbach in 1916 and was only proved in 1985 by de Branges. Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions.

A set $\mathcal{D}$ in the complex plane is called *convex* if for every pair of points w_1 and w_2 lying in the interior of $\mathcal{D}$, the line segment joining w_1 and w_2 also lies in the interior of $\mathcal{D}$, i.e.

$$tw_1 + (1-t)w_2 \ \in \mathcal{D} \quad \text{for} \quad 0 \le t \le 1.$$

If a function $f \in \mathcal{A}$ maps U onto a convex domain, then $f(z)$ is called a convex function. The class of all convex functions in $\mathcal{A}$ is denoted by K. An analytic description of the class K is given by

$$K := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

Let w_0 be an interior point of $\mathcal{D}$. A set $\mathcal{D}$ in the complex plane is called *starlike* with respect to w_0 if the line segment joining w_0 to every other point $w \in \mathcal{D}$ lies in the interior of $\mathcal{D}$, i.e.

$$(1-t)w + tw_0 \ \in \mathcal{D} \quad \text{for} \quad 0 \le t \le 1.$$

If a function $f \in \mathcal{A}$ maps U onto a domain starlike, then $f(z)$ is called a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically,

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

These two classes K and $\mathcal{S}^*$ and several other classes such as the classes of uniformly convex functions, starlike functions of order α, and strongly starlike functions investigated in geometric function theory are characterized by either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a given region in the right half-plane.

Ma and Minda [**28**] showed that many of these properties can be obtained by a unified method. For this purpose, they introduced the classes $K(\varphi)$ and $S^*(\varphi)$ of functions $f(z) \in \mathcal{A}$ for some analytic function $\varphi(z)$ with positive real part on U with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk U onto a region starlike with respect to 1, symmetric with respect to the real axis, satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$$

and

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (z \in U).$$

respectively. Here the symbol $\prec$ denotes subordination.

A function f is said to be subordinate to F in U, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in U, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

CHAPTER 2

CONVOLUTION AND DIFFERENTIAL SUBORDINATION OF MULTIVALENT FUNCTIONS

2.1. MOTIVATION AND PRELIMINARIES

Let $\mathcal{A}_p$ $(p \in \mathcal{N})$ be the class of all analytic functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots$$

with $\mathcal{A} := \mathcal{A}_1$. For two functions

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots \quad \text{and} \quad g(z) = z^p + b_{p+1}z^{p+1} + \dots$$

in $\mathcal{A}_p$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

For univalent functions, the well-known Alexander theorem [**3**], states that $f \in K$ if and only if $zf'(z) \in \mathcal{S}^*$. Since $zf'(z) = f(z) * (z/(1-z)^2)$, it follows that f is convex if and only if $f * g$ is starlike for $g(z) = z/(1-z)^2$. Moreover, since $f(z) = f(z) * (z/(1-z))$, the investigation of the classes of convex and starlike functions can be unified by considering the class of functions f for which $f * g$ is starlike for a fixed function g. These ideas motivated the investigation of the class of functions f for which

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z)$$

where g is a fixed function in $\mathcal{A}$ and h is a convex function with positive real part. Shanmugam [**55**] introduced this class and several other related classes, and investigated inclusion and convolution properties by using the convex hull method [**10, 54, 53**] and the method of differential subordination.

Motivated by the investigation of Shanmugam [**55**], Ravichandran [**45**] and Ali *et al.* [**4**] (see also [**6, 39, 38, 40**]), we introduce the following classes of multivalent functions. Throughout this chapter, the function $g \in \mathcal{A}_p$ is a fixed function and, unless otherwise mentioned, the function h is assumed to be a fixed convex univalent function with positive real part and $h(0) = 1$.

DEFINITION 2.1.1. The class $\mathcal{S}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*f)(z)}{z^p} \neq 0$ in U and satisfying the subordination

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z).$$

Similarly, $\mathcal{K}_{p,g}(h)$ is the class of functions $f \in \mathcal{A}_p$ satisfying $\frac{(g*f)'(z)}{z^{p-1}} \neq 0$ in U and

$$\frac{1}{p}\left[1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right] \prec h(z).$$

With $g(z) = z^p/(1-z)$, the class $\mathcal{S}_{p,g}(h) =: \mathcal{S}_p^*(h)$ and $\mathcal{K}_{p,g}(h) =: \mathcal{K}_p(h)$ consists respectively of all p-valent starlike and convex functions satisfying the respective subordinations

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec h(z), \text{ and } \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec h(z).$$

For these two classes, several interesting properties including distortion, growth and rotation inequalities as well as convolution properties have been investigated by Ali *et al.* [**4**]. Note that the two classes $\mathcal{S}_p^*(h)$ and $\mathcal{S}_{p,g}(h)$ are closely related; in fact, $f \in \mathcal{S}_{p,g}(h)$ if and only if $f * g \in \mathcal{S}_p^*(h)$. Similarly, $f \in \mathcal{K}_{p,g}(h)$ if and only if $f * g \in \mathcal{K}_p(h)$.

DEFINITION 2.1.2. The class $\mathcal{C}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*\psi)(z)}{z^p} \neq 0$ in U for some $\psi \in \mathcal{S}_{p,g}(h)$ and satisfying

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi)(z)} \prec h(z).$$

DEFINITION 2.1.3. For any real number γ, the class $\mathcal{K}^{\gamma}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*f)(z)}{z^p} \neq 0$ and $\frac{(g*f)'(z)}{z^{p-1}} \neq 0$ in U, and satisfying the subordination

$$\frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right]+\frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right] \prec h(z).$$

DEFINITION 2.1.4. Let $\mathcal{Q}_{p,g}(h)$ denote the class of functions $f \in \mathcal{A}_p$ such that $\frac{(g*\delta)'(z)}{z^{p-1}} \neq 0$ in U for some $\delta \in \mathcal{K}_{p,g}(h)$ and satisfying the subordination

$$\frac{1}{p}\frac{[z(g*f)'(z)]'}{(g*\delta)'(z)} \prec h(z).$$

Polya-Schoenberg [**41**] conjectured that the class K of convex functions is preserved under convolution with convex functions:

$$f, g \in K \Rightarrow f*g \in K.$$

In 1973, Ruscheweyh and Sheil-Small [**54**] proved the Polya-Schoenberg conjecture. In fact, they proved that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. For an interesting development on these ideas, see Ruscheweyh [**53**] (and also Duren [**16**, pp. 246–258], as well as Goodman [**19**, pp. 129-130]). Using the techniques developed in Ruscheweyh [**53**], several authors [**4, 7, 8, 9, 10, 21, 23, 32, 33, 34, 39, 38, 40, 47, 45, 51, 55, 57, 58**] have proved that their classes are closed under convolution with convex (and other related) functions.

In this chapter, convolution properties as well as inclusion and related properties are investigated for the general classes of p-valent functions defined above. These classes are of course extensions of the classes of convex, starlike, close-to-convex, α-convex, and quasi-convex functions. The results obtained here extend the well-known convolution properties of p-valent functions.

The following definition and results are needed to prove our main results. For $\alpha \leq 1$, the class $\mathcal{R}_\alpha$ of *prestarlike* functions of order α consists of functions $f \in \mathcal{A}$

satisfying

$$\begin{cases} f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha), & (\alpha < 1); \\ \Re\frac{f(z)}{z} > \frac{1}{2}, & (\alpha = 1) \end{cases}$$

where $\mathcal{S}^*(\alpha)$ is the class introduced by Ma and Minda [**28**].

THEOREM 2.1.1. [**53**, Theorem 2.4] *Let $\alpha \leq 1$, $f \in \mathcal{R}_\alpha$ and $g \in S^*(\alpha)$. Then for any analytic function $H \in \mathcal{H}(U)$,*

$$\frac{f * Hg}{f * g}(U) \subset \overline{co}(H(U))$$

where $\overline{co}(H(U))$ denotes the closed convex hull of $H(U)$.

THEOREM 2.1.2. [**17, 31**] *Let $\beta, \nu \in \mathcal{C}$, and $h \in \mathcal{H}(U)$ be convex univalent in U, with $\Re(\beta h(z) + \nu) > 0$. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z).$$

THEOREM 2.1.3. [**31**, Theorem 3.2b] *Let $h \in \mathcal{H}(U)$ be convex univalent in U with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \nu} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$. If $p(z) = a + a_1 z + \cdots$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

THEOREM 2.1.4. [**31**, Theorem 3.1a] *Let h be convex in U and let $P : U \to \mathcal{C}$, with $\Re P(z) > 0$. If p is analytic in U, then*

$$p(z) + P(z)zp'(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z).$$

We will also be using the following convolution properties.

(i) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$(f * g)(z) = (g * f)(z).$$

PROOF. For f and g as given, we have

$$\begin{aligned}(f * g)(z) &= z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \\ &= z^p + \sum_{n=p+1}^{\infty} b_n a_n z^n \\ &= (g * f)(z). \end{aligned}$$

□

(ii) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$\frac{1}{p} z (g * f)'(z) = (g * \frac{1}{p} z f')(z).$$

PROOF. For f of the given form, we have

$$\frac{1}{p} z f'(z) = z^p + \sum_{n=p+1}^{\infty} \frac{1}{p} n a_n z^n$$

and hence

$$\begin{aligned}(g * \frac{1}{p} z f')(z) &= z^p + \sum_{n=p+1}^{\infty} \frac{1}{p} n a_n b_n z^n \\ &= \frac{1}{p} z \left(p z^{p-1} + \sum_{n=p+1}^{\infty} n a_n b_n z^{n-1} \right) \\ &= \frac{1}{p} z (g * f)'(z). \end{aligned}$$

□

(iii) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$\frac{(g * f)(z)}{z^{p-1}} = (\frac{g}{z^{p-1}} * \frac{f}{z^{p-1}})(z).$$

PROOF. For

$$(g * f)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n,$$

we observe that

$$\begin{aligned}\frac{(g * f)(z)}{z^{p-1}} &= \frac{1}{z^{p-1}} \left(z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \right) \\ &= z + \frac{1}{z^{p-1}} \sum_{n=p+1}^{\infty} a_n b_n z^n \\ &= \frac{g(z)}{z^{p-1}} * \frac{f(z)}{z^{p-1}}.\end{aligned}$$

□

2.2. INCLUSION AND CONVOLUTION THEOREMS

Every convex univalent function is starlike or equivalently $K \subset \mathcal{S}^*$, and Alexander's theorem gives $f \in K$ if and only if $zf' \in \mathcal{S}^*$. These properties remain valid even for multivalent functions.

THEOREM 2.2.1. *Let g be a fixed function in $\mathcal{A}_p$ and h be a convex univalent function with positive real part and $h(0) = 1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$,

(ii) $f \in \mathcal{K}_{p,g}(h)$ *if and only if* $\frac{1}{p} zf' \in \mathcal{S}_{p,g}(h)$.

PROOF. (i) Since $(f * g)(z)/z^p \neq 0$, the function q defined by

$$q(z) = \frac{1}{p} \frac{z(g * f)'(z)}{(g * f)(z)},$$

is analytic in U. By some computations we have,

$$\begin{aligned}\frac{zq'(z)}{q(z)} &= 1 + \frac{z(g * f)''(z)}{(g * f)'(z)} - \frac{z(g * f)'(z)}{(g * f)(z)} \\ &= 1 + \frac{z(g * f)''(z)}{(g * f)'(z)} - pq(z).\end{aligned}$$

Equivalently, we have

$$pq(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{z(g * f)''(z)}{(g * f)'(z)}$$

and satisfies

$$q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}=\frac{1}{p}\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right). \tag{2.2.1}$$

If $f\in\mathcal{K}_{p,g}(h)$, the right-hand side of (2.2.1) is subordinate to h. It follows from Theorem 2.1.2 that $q(z)\prec h(z)$, and thus $\mathcal{K}_{p,g}(h)\subseteq\mathcal{S}_{p,g}(h)$.

(ii) Since

$$\begin{aligned}\frac{1}{p}\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right)&=\frac{1}{p}\left[\frac{(g*f)'(z)+z(g*f)''(z)}{(g*f)'(z)}\right]\\&=\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{(g*zf')'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{\frac{1}{p}z}{\frac{1}{p}z}\frac{(g*zf')'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}zf')(z)},\end{aligned}$$

it follows that $f\in\mathcal{K}_{p,g}(h)$ if and only if $\frac{1}{p}zf'\in\mathcal{S}_{p,g}(h)$. □

Suppose that the differential equation

$$q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}=h(z)$$

has a univalent solution q that satisfies $q(z)\prec h(z)$. If $f\in\mathcal{K}_{p,g}(h)$, then from Theorem 2.1.3 and (2.2.1), it follows that $f\in\mathcal{S}_{p,g}(q)$, or equivalently $\mathcal{K}_{p,g}(h)\subset\mathcal{S}_{p,g}(q)$.

THEOREM 2.2.2. *Let h be a convex univalent function satisfying the condition*

$$\Re h(z)>1-\frac{1-\alpha}{p}\quad(0\le\alpha<1), \tag{2.2.2}$$

and $\phi\in\mathcal{A}_p$ with $\phi/z^{p-1}\in\mathcal{R}_\alpha$. If $f\in\mathcal{S}_{p,g}(h)$, then $\phi*f\in\mathcal{S}_{p,g}(h)$.

PROOF. For $f\in\mathcal{S}_{p,g}(h)$, let the function H be defined by

$$H(z):=\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

Then H is analytic in U and $H(z) \prec h(z)$. Also let $\Phi(z) := \phi(z)/z^{p-1}$ belongs to $\mathcal{R}_\alpha$. We now show that the function $G(z) := (f * g)(z)/z^{p-1}$ belongs to $\mathcal{S}^*(\alpha)$.

Since $f \in \mathcal{S}_{p,g}(h)$, and h is a convex univalent function satisfying (2.2.2), it follows that

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > 1 - \frac{1-\alpha}{p},$$

and hence

$$\begin{aligned}\Re\frac{zG'(z)}{G(z)} &= \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) - p + 1 \\ &> p - 1 + \alpha - p + 1 \\ &> \alpha.\end{aligned}$$

Thus $G \in \mathcal{S}^*(\alpha)$. Since $\Phi \in \mathcal{R}_\alpha$, $G \in \mathcal{S}^*(\alpha)$, and h is convex, an application of Theorem 2.1.1 shows that

$$\frac{(\Phi * GH)(z)}{(\Phi * G)(z)} \prec h(z). \tag{2.2.3}$$

We have

$$\begin{aligned}\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} &= \frac{\phi(z) * \frac{1}{p}z(g*f)'(z)}{\phi(z) * (g*f)(z)} \\ &= \frac{\frac{\phi(z)}{z^{p-1}} * \frac{\frac{1}{p}z(g*f)'(z)}{z^{p-1}}}{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}} \\ &= \frac{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}} \\ &= \frac{(\Phi * GH)(z)}{(\Phi * G)(z)}.\end{aligned}$$

Thus the subordination (2.2.3) gives

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} \prec h(z),$$

which proves $\phi * f \in \mathcal{S}_{p,g}(h)$. □

Corollary 2.2.1. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. Then $\mathcal{S}_{p,g}(h) \subseteq \mathcal{S}_{p,\phi*g}(h)$.*

Proof. If $f \in \mathcal{S}_{p,g}(h)$, Theorem 2.2.2 yields $f * \phi \in \mathcal{S}_{p,g}(h)$, that is $f * \phi * g \in \mathcal{S}_p^*(h)$. Hence $f \in \mathcal{S}_{p,\phi * g}(h)$. □

In particular, when $g(z) = z^p/(1-z)$, the following corollary is obtained.

Corollary 2.2.2. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{S}_p^*(h)$, then $f \in \mathcal{S}_{p,\phi}^*(h)$.*

Corollary 2.2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{K}_{p,g}(h)$, then $f * \phi \in \mathcal{K}_{p,g}(h)$ and $\mathcal{K}_{p,g}(h) \subseteq \mathcal{K}_{p,\phi * g}(h)$.*

Proof. If $f \in \mathcal{K}_{p,g}(h)$, it follows from Theorem 2.2.1(ii) and Theorem 2.2.2 that $\frac{1}{p}(zf' * \phi) \in \mathcal{S}_{p,g}(h)$. Hence $f * \phi \in \mathcal{K}_{p,g}(h)$. The second part follows from Corollary 2.2.1. □

Theorem 2.2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{C}_{p,g}(h)$ with respect to $\psi \in \mathcal{S}_{p,g}(h)$, then $\phi * f \in \mathcal{C}_{p,g}(h)$ with respect to $\phi * \psi \in \mathcal{S}_{p,g}(h)$.*

Proof. As in the proof of Theorem 2.2.2, define the functions H, Φ and G by

$$H(z) := \frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi)(z)}, \quad \Phi(z) := \frac{\phi(z)}{z^{p-1}}, \quad \text{and} \quad G(z) := \frac{(\psi * g)(z)}{z^{p-1}}.$$

Then $\Phi \in \mathcal{R}_\alpha$ and $G \in \mathcal{S}^*(\alpha)$. An application of Theorem 2.1.1 shows that the quantity $\dfrac{(\Phi * GH)(z)}{(\Phi * G)(z)}$ lies in the closed convex hull of $H(U)$. Since h is convex and $H \prec h$, it follows that

$$\frac{(\Phi * GH)(z)}{(\Phi * G)(z)} \prec h(z). \tag{2.2.4}$$

Observe that

$$\begin{aligned}
\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*\psi)(z)} &= \frac{\phi(z)*\frac{1}{p}z(g*f)'(z)}{\phi(z)*(g*\psi)(z)} \\
&= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{\frac{1}{p}z(g*f)'(z)}{z^{p-1}}}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}} \\
&= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}} \\
&= \frac{(\Phi*GH)(z)}{(\Phi*G)(z)}.
\end{aligned}$$

Thus, the subordination (2.2.4) shows that $\phi*f\in\mathcal{C}_{p,g}(h)$ with respect to $\phi*\psi\in\mathcal{S}_{p,g}(h)$. □

COROLLARY 2.2.4. *If h and ϕ satisfy the conditions of Theorem 2.2.2, then $\mathcal{C}_{p,g}(h)\subseteq\mathcal{C}_{p,\phi*g}(h)$.*

PROOF. From Theorem 2.2.3, for a function $f\in\mathcal{C}_{p,g}(h)$ with respect to $\psi\in\mathcal{S}_{p,g}(h)$, we have

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*\psi)(z)}\prec h(z).$$

Thus, $f\in\mathcal{C}_{p,\phi*g}(h)$, and hence $\mathcal{C}_{p,g}(h)\subseteq\mathcal{C}_{p,\phi*g}(h)$. □

THEOREM 2.2.4. *Let h be a convex univalent function with positive real part and $h(0)=1$. Then*

(i) $\mathcal{K}^{\gamma}_{p,g}(h)\subseteq\mathcal{S}_{p,g}(h)$ *for* $\gamma>0$,

(ii) $\mathcal{K}^{\gamma}_{p,g}(h)\subseteq\mathcal{K}^{\beta}_{p,g}(h)$ *for* $\gamma>\beta\geq 0$,
where $\mathcal{K}^{\gamma}_{p,g}(h)$ is defined in Definition 2.1.3.

PROOF. (i) Let

$$J_{p,g}(\gamma;f(z)):=\frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right]+\frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right]$$

and the function $q(z)$ be defined by

$$q(z) := \frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

Note that

$$\begin{aligned} J_{p,g}(\gamma; f(z)) &= \frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right] + \frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right] \\ &= \gamma\left(q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}\right)+(1-\gamma)q(z) \\ &= q(z)+\frac{\gamma zq'(z)}{pq(z)}. \qquad (2.2.5) \end{aligned}$$

Let $f \in \mathcal{K}^{\gamma}_{p,g}(h)$, so $J_{p,g}(\gamma; f(z)) \prec h(z)$. Now an application of Theorem 2.1.2 shows that $q(z) \prec h(z)$. Hence $f \in \mathcal{S}_{p,g}(h)$.

(ii) The case $\beta = 0$ is contained in (i), so we assume $\beta > 0$. We define $q(z)$ as in (i), then (2.2.5) yields

$$J_{p,g}(\beta; f(z)) = q(z) + \frac{\beta}{p}\frac{zq'(z)}{q(z)}. \qquad (2.2.6)$$

Since

$$J_{p,g}(\gamma; f(z)) = q(z) + \frac{\gamma zq'(z)}{pq(z)},$$

we have

$$\frac{1}{\gamma}\left(J_{p,g}(\gamma; f(z)) - q(z)\right) = \frac{1}{p}\frac{zq'(z)}{q(z)}. \qquad (2.2.7)$$

Substituting (2.2.7) in (2.2.6), we have

$$J_{p,g}(\beta; f(z)) = (1-\frac{\beta}{\gamma})\frac{z(g*f)'(z)}{p(g*f)(z)} + \frac{\beta}{\gamma}J_{p,g}(\gamma; f(z))$$

From part (i),

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z)$$

and

$$J_{p,g}(\gamma; f(z)) \prec h(z).$$

Since $J_{p,g}(\beta; f(z))$ is a convex combination of points in $h(U)$, and h is convex, it follows that $J_{p,g}(\beta; f(z)) \prec h(z)$, proving that $f \in \mathcal{K}^{\beta}_{p,g}(h)$. □

Theorem 2.2.5. *Let h be a convex univalent function with positive real part and $h(0)=1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h) \subseteq \mathcal{C}_{p,g}(h)$,

(ii) $f \in \mathcal{Q}_{p,g}(h)$ *if and only if* $\frac{1}{p} z f' \in \mathcal{C}_{p,g}(h)$.

Proof. (i) By taking $f = \delta$, it follows from the definition that $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h)$. To prove the middle inclusion, let

$$q(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)(z)}.$$

By logarithmic differentiation and multiplication of z, we have

$$\frac{zq'(z)}{q(z)} = 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} - \frac{z(g*\delta)'(z)}{(g*\delta)(z)}.$$

Rewriting the equation,

$$zq'(z) = q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z) - \frac{z(g*\delta)'(z)}{(g*\delta)(z)}q(z)$$

$$\frac{z(g*\delta)'(z)}{(g*\delta)(z)}q(z) + zq'(z) = q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z)$$

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{(g*\delta)(z)}{z(g*\delta)'(z)}q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z)\cdot\frac{(g*\delta)(z)}{z(g*\delta)'(z)}.$$

Substituting $q(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)(z)}$ on the right-hand side of the above equation, we have

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)'(z)} + \frac{1}{p}\frac{z(g*f)''(z)}{(g*\delta)'(z)}.$$

The above computations shows that

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\delta)'(z)}. \tag{2.2.8}$$

If $f \in \mathcal{Q}_{p,g}(h)$, then there exists a function $\delta \in \mathcal{K}_{p,g}(h)$ such that the expression on the right-hand side of (2.2.8) is subordinate to $h(z)$. Also $\delta \in \mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$ implies $\Re\frac{z(g*\delta)'(z)}{(g*\delta)} > 0$. Hence, an application of Theorem 2.1.4 to (2.2.8) yields $q(z) \prec h(z)$. This shows that $f \in \mathcal{C}_{p,g}(h)$.

(ii) It is easy to see that

$$\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\delta)'(z)} = \frac{1}{p}\frac{(g*zf')'(z)}{(g*\delta)'(z)}\cdot\frac{\frac{1}{p}z}{\frac{1}{p}z}$$

$$(2.2.9) \qquad = \frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}z\delta')(z)}.$$

Now if $f\in\mathcal{Q}_{p,g}(h)$ with respect to a function $\delta\in\mathcal{K}_{p,g}(h)$, then the expression on the left-hand side of (2.2.9) is subordinate to $h(z)$. Now by Theorem 2.2.1(ii) and definition of $\mathcal{C}_{p,g}(h)$, we get $\frac{1}{p}zf'\in\mathcal{C}_{p,g}(h)$.

Conversely, if $\frac{1}{p}zf'\in\mathcal{C}_{p,g}(h)$, then there exists a function $\delta_1\in\mathcal{S}_{p,g}(h)$ such that $\frac{1}{p}z\delta'=\delta_1$. The expression on the right-hand side of (2.2.9) is subordinate to $h(z)$ and thus $f\in\mathcal{Q}_{p,g}(h)$. □

COROLLARY 2.2.5. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f\in\mathcal{Q}_{p,g}(h)$, then $\phi*f\in\mathcal{Q}_{p,g}(h)$.*

PROOF. If $f\in\mathcal{Q}_{p,g}(h)$, then by Theorem 2.2.5(ii), $\frac{1}{p}zf'\in\mathcal{C}_{p,g}(h)$. Theorem 2.2.3 shows that $\frac{1}{p}z(\phi*f)'\in\mathcal{C}_{p,g}(h)$, and by Theorem 2.2.5(ii), we have $\phi*f\in\mathcal{Q}_{p,g}(h)$. □

COROLLARY 2.2.6. *If h and ϕ satisfy the conditions of Theorem 2.2.2, then $\mathcal{Q}_{p,g}(h)\subseteq\mathcal{Q}_{p,\phi*g}$.*

PROOF. If $f\in\mathcal{Q}_{p,g}(h)$, Corollary 2.2.5 yields $f*\phi\in\mathcal{Q}_{p,g}(h)$ with respect to $\phi*\delta\in\mathcal{K}_{p,g}(h)$. The subordination

$$\frac{1}{p}\frac{[z(g*\phi*f)'(z)]'}{(g*\phi*\delta)'(z)}\prec h(z)$$

gives $f\in\mathcal{Q}_{p,g*\phi}$. Therefore, $\mathcal{Q}_{p,g}(h)\subseteq\mathcal{Q}_{p,g*\phi}$. □

A function is prestarlike of order 0 if $f(z)*(z/(1-z)^2)$ is starlike, or equivalently if f is convex. Thus, the class of prestarlike functions of order 0 is the class of convex functions, and therefore the results obtained in this chapter contains the results of Shanmugam [**55**] for the special case $p = 1$ and $\alpha = 0$.

EXAMPLE 2.2.1. Let $p = 1$, $g(z) = z/(1 - z)$, and $\alpha = 0$. For $h(z) = (1 + z)/(1 - z)$, Theorem 2.2.1 reduces to the following: $K \subseteq \mathcal{S}^*$ and $f \in K \Leftrightarrow zf' \in \mathcal{S}^*$. Also Theorem 2.2.2 reduces to $f \in \mathcal{S}^*, g \in K \Rightarrow f * g \in \mathcal{S}^*$, and Corollary 2.23 shows that the class of convex functions is closed under convolution with convex functions.

For

$$h(z) = 1 + \frac{2}{\pi^2}\left[\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2,$$

the results obtained imply that $\mathcal{UCV} \subseteq \mathcal{S}_p$ and $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{S}_p$, where $\mathcal{UCV}$ and $\mathcal{S}_p$ are the classes of uniformly convex functions and parabolic starlike functions [**51, 50**]. It also follows as special cases that the classes $\mathcal{S}_p$ and $\mathcal{UCV}$ are closed under convolution with convex functions.

CHAPTER 3

THE FEKETE-SZEGÖ COEFFICIENT FUNCTIONAL FOR TRANSFORMS OF ANALYTIC FUNCTIONS

3.1. MOTIVATION AND PRELIMINARIES

For a univalent function in the class $\mathcal{A}$, it is well known that the n-th coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The Fekete-Szegö coefficient functional also arise in the investigation of univalency of analytic functions. Several authors have investigated the Fekete-Szegö functional for functions in various subclasses of univalent and multivalent functions [**1, 7, 5, 2, 11, 14, 13, 15, 22, 25, 35, 36, 44, 46, 56**], and more recently by Choi, Kim, and Sugawa [**12**].

Ma and Minda [**28**] gave a unified treatment of various subclasses consisting of starlike and convex functions for which either the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. In fact, they considered the analytic function φ with positive real part in the unit disk U, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis.

The unified class $S^*(\varphi)$ introduced by Ma and Minda [**28**] consists of starlike functions and they also investigated the corresponding class $K(\varphi)$ of convex functions, for $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (z \in U)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$$

respectively. Ma and Minda [**28**] obtained subordination results, distortion, growth and rotation theorems. They also obtained estimates for the first few coefficients and determined bounds for the associated Fekete-Szegö functional. A function $f \in S^*(\varphi)$ is said to be starlike function with respect to φ, and a function $f \in K(\varphi)$ is a convex function with respect to φ.

The unified treatment of various subclasses of starlike and convex functions by Ma and Minda [**28**] motivates one to consider similar classes defined by subordination. In this chapter, we consider the following classes of functions which have been defined earlier by several authors in [**27**], [**37**], [**42**] [**43**], [**48**].

$$\begin{aligned}
R_b(\varphi) &:= \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\}, \\
S^*(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}, \\
L(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\}, \\
M(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},
\end{aligned}$$

where $b \in \mathcal{C} \setminus \{0\}$, and $\alpha \geq 0$. Some coefficient problems for functions f belonging to certain classes of p-valent functions were investigated in [**5**].

For a univalent function $f(z)$ of the form

(3.1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

the k-th root transform is defined by

(3.1.2) $$F(z) := [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}.$$

In Section 3.2, sharp bounds for the Fekete-Szegö coefficient functional $\left|b_{2k+1} - \mu b_{k+1}^2\right|$ associated with the k-th root transform of the function f belonging to the above mentioned classes are derived. In Section 3.3, a similar problem is investigated for functions G where $G(z) := z/f(z)$ and the function f belongs to the above mentioned classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. The following two lemmas regarding the coefficients of functions in Ω are needed to prove our main results. Lemma 3.1.1 is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [**28**].

LEMMA 3.1.1. [**5**] *If $w \in \Omega$ and*

(3.1.3) $$w(z) := w_1 z + w_2 z^2 + \cdots \quad (z \in U),$$

then

$$|w_2 - t w_1^2| \leq \begin{cases} -t & \text{if } t \leq -1 \\ 1 & \text{if } -1 \leq t \leq 1 \\ t & \text{if } t \geq 1. \end{cases}$$

When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ or one of its rotations.

LEMMA 3.1.2. [**22**] *If $w \in \Omega$, then*

$$|w_2 - t w_1^2| \leq \max\{1;\ |t|\},$$

for any complex number t. The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

3.2. COEFFICIENT BOUNDS FOR THE k-TH ROOT TRANSFORMATION

In the first theorem below, the bound for the coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k-th root transformation of starlike functions with respect to φ is given. Notice that the classes $S^*(\alpha,\varphi)$, $L(\alpha,\varphi)$ and $M(\alpha,\varphi)$ reduce to the class $S^*(\varphi)$ for appropriate choice of the parameters. Although Theorem 3.2.1 is contained in the corresponding results for the classes $S^*(\alpha,\varphi)$, $L(\alpha,\varphi)$ and $M(\alpha,\varphi)$, it is stated and proved separately here because of its importance in its own right as well as to illustrate the main ideas.

THEOREM 3.2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *and*

$$\sigma_1 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right], \quad \sigma_2 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right].$$

If f given by (3.1.1) belongs to $S^*(\varphi)$, and F is the k-th root transformation of f given by (3.1.2), then

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \dfrac{B_1^2}{2k^2}(1-2\mu) + \dfrac{B_2}{2k}, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\dfrac{B_1^2}{2k^2}(1-2\mu) - \dfrac{B_2}{2k}, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k} \max\left\{1; \left|\frac{B_1}{k}(1-2\mu) + \frac{B_2}{B_1}\right|\right\}.$$

PROOF. If $f \in S^*(\varphi)$, then there is an analytic function $w \in \Omega$ of the form (3.1.3) such that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{3.2.1}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{z(1+2a_2z+3a_3z^2+\dots)}{z+a_2z^2+a_3z^3+\dots} = \frac{1+2a_2z+3a_3z^2+\dots}{1+a_2z+a_3z^2+\dots}$$
$$= (1+2a_2z+3a_3z^2+\dots)[1-(a_2z+a_3z^2+\dots)+(a_2z+a_3z^2+\dots)^2-\dots]$$
$$= 1+a_2z+(-a_2^2+2a_3)z^2+(3a_4-3a_2a_3+a_2^3)z^3+\dots$$

and

$$\varphi(w(z)) = 1+B_1(w_1z+w_2z^2+\dots)+B_2(w_1z+w_2z^2+\dots)^2+\dots$$
$$= 1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\dots,$$

it follows from (3.2.1) that

(3.2.2) $$a_2 = B_1w_1$$

and

(3.2.3) $$a_3 = \frac{1}{2}[B_1w_2+(B_2+B_1^2)w_1^2].$$

For a function f given by (3.1.1), we have

$$[f(z^k)]^{1/k} = [z^k+a_2z^{2k}+a_3z^{3k}+\dots]^{1/k}$$
$$= [z^k(1+a_2z^k+a_3z^{2k}+\dots)]^{1/k}$$
$$= z\left[1+\frac{1}{k}(a_2z^k+a_3z^{2k}+\dots)+\frac{1-k}{2k^2}(a_2z^k+a_3z^{2k}+\dots)^2+\dots\right]$$

(3.2.4) $$= z+\frac{1}{k}a_2z^{k+1}+\left(\frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1}+\cdots.$$

The equations (3.1.2) and (3.2.4) yield

(3.2.5) $$b_{k+1} = \frac{1}{k}a_2,$$

and

(3.2.6) $$b_{2k+1} = \frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2.$$

On substituting for a_2 and a_3 in (3.2.5) and (3.2.6) from (3.2.2) and (3.2.3), it follows that

$$b_{k+1} = \frac{B_1 w_1}{k}$$

and

$$b_{2k+1} = \frac{1}{2k}\left[B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{k}\right],$$

and hence

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k}\left\{w_2 - \left[-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

The first half of the result is established by an application of Lemma 3.1.1.

If $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq -1$, then

$$\mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \quad (\mu \leq \sigma_1),$$

and Lemma 3.1.1 gives

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1^2}{2k^2}(1-2\mu) + \frac{B_2}{2k}.$$

For $-1 \leq -\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq 1$, we have

$$\frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \leq \mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\sigma_1 \leq \mu \leq \sigma_2),$$

and Lemma 3.1.1 yields

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k}.$$

For $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \geq 1$, we have

$$\mu \geq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\mu \geq \sigma_2),$$

and it follows from Lemma 3.1.1 that

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq -\frac{B_1^2}{2k^2}(1-2\mu) - \frac{B_2}{2k}.$$

The second half of the result follows by an application of Lemma 3.1.2:

$$\begin{aligned}\left|b_{2k+1}-\mu b_{k+1}^2\right| &= \frac{B_1}{2k}\left|w_2-\left[-\frac{B_1}{k}(1-2\mu)-\frac{B_2}{B_1}\right]w_1^2\right| \\ &\leq \frac{B_1}{2k}\max\left\{1;\ \left|\frac{B_1}{k}(1-2\mu)+\frac{B_2}{B_1}\right|\right\}.\end{aligned}$$ □

REMARK 3.2.1.

(1) In view of the Alexander result **[3]** that $f \in K(\varphi)$ if and only if $zf' \in S^*(\varphi)$, the estimate for $\left|b_{2k+1}-\mu b_{k+1}^2\right|$ for a function in $K(\varphi)$ can be obtained from the corresponding estimates in Theorem 3.2.1 for functions in $S^*(\varphi)$.

(2) For $k = 1$, the k-th root transformation of f reduces to the given function f itself. Thus, the estimate given in equation (3.2.1) of Theorem 3.2.1 is an extension of the corresponding result for the Fekete-Szegö coefficient functional corresponding to functions starlike with respect to φ. Similar remark applies to the other results in this section.

The well-known Noshiro-Warschawski theorem states that a function $f \in \mathcal{A}$ with positive derivative in U is univalent. The class $R_b(\varphi)$ of functions defined in terms of the subordination of the derivative f' is closely associated with the class of functions with positive real part. The bound for the Fekete-Szegö functional corresponding to the k-th root transformation of functions in $R_b(\varphi)$ is given in Theorem 3.2.2.

THEOREM 3.2.2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* f *given by (3.1.1) belongs to* $R_b(\varphi)$*, and* F *is the* k*-th root transformation of* f *given by (3.1.2), then*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{|b|B_1}{3k}\max\left\{1;\left|\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right|\right\}.$$

PROOF. If $f \in R_b(\varphi)$, then there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ such that

$$1+\frac{1}{b}\left(f'(z)-1\right)=\varphi(w(z)). \tag{3.2.7}$$