UNIVERSITI SAINS MALAYSIA

First Semester Examinations
1997/98 Academic Session

September 1997

<u>ZCT 211/2 - Vector Analysis</u>

Time: [2 hours]

Please make sure that this examination paper consists of FOUR printed pages before you commence the examination.

Answer EIGHT questions only. Candidates may choose to answer all questions in the Malay Language. If candidates choose to answer in the English Language, it is compulsory to answer at least one question in the Malay Language.

1. Two vectors in component form are $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$. Write down the expressions in terms of the components for $\vec{A}.\vec{B}$ and $\vec{A} \times \vec{B}$.

(20/100)

Given $\vec{A} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{B} = \vec{i} - 2\vec{j} + \vec{k}$ evaluate $|\vec{A}|$, $|\vec{B}|$, $\vec{A}.\vec{B}$ and $\vec{A} \times \vec{B}$.

(50/100)

Hence find the angle between $\vec{A}$ and $\vec{B}$.

(30/100)

2. Write down the expression for $\vec{A}.(\vec{B} \times \vec{C})$ in terms of the components of $\vec{A}$, $\vec{B}$ and $\vec{C}$. What is the geometrical meaning of $\vec{A}.(\vec{B} \times \vec{C})$?

(20/100)

State the expansion of $\vec{A} \times (\vec{B} \times \vec{C})$ in terms of $\vec{B}$ and $\vec{C}$.

(20/100)

For $\vec{A} = \vec{i} + 3\vec{j} + 2\vec{k}$, $\vec{B} = 4\vec{i} - \vec{k}$ and $\vec{C} = 5\vec{i} + 2\vec{j} - 3\vec{k}$ evaluate $\vec{A}.(\vec{B} \times \vec{C})$ and $\vec{A} \times (\vec{B} \times \vec{C})$, expressing the latter in component form.

(60/100)

3. $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ is a vector field and ϕ is a scalar field. State the definitions of $\nabla.\vec{A}$, $\nabla \times \vec{A}$, $\nabla\phi$ and $\nabla^2\phi$.

(20/100)

Find $\nabla\phi$ and $\nabla^2\phi$ for $\phi = r^3 = (x^2 + y^2 + z^2)^{3/2}$.

(60/100)

Confirm by explicit evaluation that $\nabla \times (\nabla\phi) = 0$.

(20/100)

4. Two vector fields are $\vec{A} = xz\vec{i} + (y-z)^2\vec{j} + 2xyz\vec{k}$ and $\vec{B} = 2y\vec{i} + 4z\vec{j} + x^2z^2\vec{k}$. Evaluate

$\nabla \times \vec{A}$,

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$\nabla \times \vec{B}$,

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$\vec{A} \times \vec{B}$ and

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$\nabla \times (\vec{A} \times \vec{B})$.

(30/100)

5. A particle moves so that at time t its position is given by $\vec{r}(t) = \alpha t\vec{i} + \frac{1}{2}\alpha t\vec{j} + (h - \frac{1}{2}\beta t^2)\vec{k}$. Find its velocity and acceleration vectors $\vec{v}(t)$ and $\vec{f}(t)$.

(40/100)

Describe the motion in words and explain precisely the initial conditions that lead to this motion. In doing this, express the initial velocity as speed and direction relative to the x axis.

(60/100)

6. Calculate the double integral $\int f(x,y)dxdy$ over the square $0 \le x \le 1$, $0 \le y \le 1$ where

(a) $f(x,y) = (x-y)^2$

(30/100)

(b) $f(x,y) = xy$

(30/100)

Calculate the volume integral $\int_V f(\vec{r})dV$ where V is the unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $f(\vec{r}) = xyz$.

(40/100)

7. State Gauss's and Stokes's theorems, defining carefully the terms you use.

(40/100)

By using Gauss's theorem or otherwise evaluate $\int_S \vec{F}.\vec{n}dS$ where

(a) $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the cube $0 \le x \le a$, $0 \le y \le a,\ 0 \le z \le a$

(30/100)

(b) $\vec{F} = \nabla\phi$ where $\phi = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + z^2$ and S is any closed surface.

(30/100)

8. State Gauss's Law relating the gravitational field $\vec{F}(\vec{r})$ to the mass density $\rho(\vec{r})$.

(20/100)

Taking a simplified model of the earth as a sphere of radius R and uniform density ρ so that its mass is $M = 4\pi\rho R^3/3$. Assuming that $\vec{F}(\vec{r})$ is radial, show that the magnitude F of $\vec{F}(\vec{r})$ at radius r is GMr/R^3 inside the earth and GM/r^2 outside the earth.

(60/100)

Draw a sketch to show the dependence of F on r.

(20/100)

9. A compressible fluid has density $n(\vec{r},t)$ and flow velocity $\vec{J}(\vec{r},t)$. Use the fact that any change in the quantity of fluid in a given volume arises from the net inflow or outflow due to $\vec{J}(\vec{r},t)$ to show that the flow satisfies the continuity equation $\frac{\partial n}{\partial t} + \nabla . \vec{J} = 0$.

(40/100)

Show further that for incompressible fluid flow $\nabla . \vec{J} = 0$.

(20/100)

A cylindrical hose pipe nozzle tapers from radius R_0 to radius R_1. Assuming that water is incompressible, find an equation giving the ratio v_1/v_0 of the velocities at these two points in terms of R_1/R_0.

(40/100)

10. The expressions for ∇V and $\nabla^2 V$ in a general orthogonal coordinate system are given below. Prove that for cylindrical polar coordinates (r, ϕ, z) $h_r = 1$, $h_\phi = r$ and $h_z = 1$.

(20/100)

Find the general form of a potential V(r) that satisfies $\nabla^2 V = 0$ and depends only on r in a cylindrical coordinate system.

(40/100)

Find the corresponding electrostatic field $\vec{E} = -\nabla V$.

(20/100)

Apply Gauss's Law of electrostatics to a uniformly charged long cylinder to explain why $\vec{E}$ has this form.

(20/100)

If $ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$ then

$$\nabla V = \left(\frac{1}{h_1}\frac{\partial V}{\partial u_1}, \frac{1}{h_2}\frac{\partial V}{\partial u_2}, \frac{1}{h_3}\frac{\partial V}{\partial u_3}\right)$$

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right]$$

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