UNIVERSITI SAINS MALAYSIA

Final Examination
Academic Session 2008/2009

April 2009

JIM 213 – Differential Equations I
[Persamaan Pembezaan I]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains TEN printed pages before you begin the examination.

Answer ALL questions.

Read the instructions carefully before answering.

Each question is worth 100 marks.

Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.

Jawab SEMUA soalan.

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

...2/-

1. (a) Find the general solution of the differential equation

$$x^2 \frac{dy}{dx} + (1+x)y^2 = 2x^3y^2.$$

(40 marks)

Given the differential equation

$$\left(1+x^2\right)\frac{dy}{dx} - \frac{4x^3y}{1-x^2} = 1, \qquad (-1 < x < 1).$$

(i) Solve the above equation to show that

$$y = \frac{k+3x-x^3}{3\left(1-x^4\right)}, \qquad (*)$$

where k is an arbitrary constant.

(ii) Find the value of k in (*) for which y tends to a finite limit as x tends to 1.

(60 marks)

2. (a) Consider the first order differential equation

$$\frac{dy}{dx} + f(x)y = g(x)y^n,$$

where $f(x)$ and g(x) are continuous functions in an interval and n is an integer. State the types of equation for the case $n = 0$, $n = 1$ and $n \neq 0, 1$. For each case, describe clearly how you would solve the differential equation.

(50 marks)

(b) If $\phi_1(x)$ is a solution of the second order differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

show that $\phi_2(x) = C\phi_1(x)$, where C is a constant, is also a solution. Can $\phi_1(x)$ and $\phi_2(x)$ form the fundamental solutions? Explain.

(50 marks)

...3/-

3. (a) Based on the method of undetermined coefficients, suggest the correct form for the particular solution of the differential equation

$$y'' + 4y = 3 \sin 2t .$$

(40 marks)

(b) Use the method of variation of parameters to find the solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = \frac{e^{6x}}{1+e^{2x}}$$

which satisfies,

$$y(0) = \frac{1}{2}\ell n 2,$$

$$y\left(\frac{1}{2}\ell n 2\right) = \frac{3}{2}\ell n 3.$$

(60 marks)

4. Consider the system of homogenous linear differential equations

$$\frac{dx}{dt} = -5x + 5y + 4z$$

$$\frac{dy}{dt} = -8x + 7y + 6z$$

$$\frac{dz}{dt} = x.$$

(a) Write the system of equations in the form

$$\frac{dX}{dt} = AX,$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and identify the matrix A.

(25 marks)

...4/-

(b) Show that the eigenvalues of A are

$$\lambda_1 = 2, \qquad \lambda_2 = i, \qquad \lambda_3 = -i.$$

Thus find their corresponding eigenvectors.

(45 marks)

(c) Find the general solution of the system.

(30 marks)

5. A function $f(t)$ is defined by

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 2t - 3, & t \ge 4 \end{cases}$$

(a) Graph the function $f(t)$ from t = 0 to t = 7.

(10 marks)

(b) Write the function $f(t)$ in term of Heaviside functions $U_a(t)$ where $U_a(t)$ is defined to be

$$U_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

(10 marks)

(c) Find the Laplace transform of $f(t)$.

(30 marks)

(d) Use the method of Laplace Transform to find the solution of the initial value problem

$$y'' + y' = f(t)$$

$$y(0) = 1$$

$$y'(0) = 0$$

where $f(t)$ is given above.

[You may use the following partial fractions

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\frac{1}{s^3(s+1)} = \frac{1}{s^3} - \frac{1}{s^s} + \frac{1}{s} - \frac{1}{s+1}$$

and the results given in Table 1].

(50 marks)

...6/-

1. (a) Cari penyelesaian am bagi persamaan pembezaan

$$x^2\frac{dy}{dx}+(1+x)y^2=2x^3y^2.$$

(40 markah)

(b) Diberi persamaan pembezaan

$$\left(1+x^2\right)\frac{dy}{dx}-\frac{4x^3y}{1-x^2}=1, \qquad (-1<x<1).$$

(i) Selesaikan persamaan di atas untuk menunjukkan bahawa

$$y=\frac{k+3x-x^3}{3\left(1-x^4\right)}, \qquad (*)$$

di mana k adalah pemalar sebarangan.

(ii) Cari nilai k dalam (*) supaya y menghampiri had terhingga apabila x menghampiri 1.

(60 markah)

2. (a) Pertimbangkan persamaan peringkat pertama

$$\frac{dy}{dx}+f(x)y=g(x)y^n,$$

di mana $f(x)$ dan $g(x)$ adalah fungsi selanjar dalam suatu selang dan n adalah suatu integer. Nyatakan jenis persamaan untuk kes $n=0$, $n=1$ dan $n\neq 0, 1$. Bagi setiap kes, huraikan dengan jelas bagaimana anda menyelesaikan persamaan tersebut.

(50 markah)

...7/-

(b) Jika $\phi_1(x)$ adalah suatu penyelesaian bagi persamaan pembezaan peringkat kedua

$$y'' + p(x)y' + q(x)y = 0,$$

tunjukkan bahawa $\phi_2(x) = C\phi_1(x)$, di mana C suatu pemalar, adalah juga suatu penyelesaian. Dapatkah $\phi_1(x)$ dan $\phi_2(x)$ membentuk penyelesaian asasi? Terangkan.

(50 markah)

3. (a) Berdasarkan kaedah koefisien belum tentu, cadangkan suatu bentuk sesuai bagi penyelesaian khusus persamaan pembezaan

$$y'' + 4y = 3\sin 2t.$$

(40 markah)

(b) Gunakan kaedah perubahan parameter untuk mencari penyelesaian persamaan pembezaan

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = \frac{e^{6x}}{1+e^{2x}}$$

yang menepati

$$y(0) = \frac{1}{2}\ell n 2,$$

$$y\left(\frac{1}{2}\ell n 2\right) = \frac{3}{2}\ell n 3.$$

(60 markah)

4. Pertimbangkan sistem persamaan pembezaan linear homogen

$$\frac{dx}{dt} = -5x + 5y + 4z$$

$$\frac{dy}{dt} = -8x + 7y + 6z$$

$$\frac{dz}{dt} = x.$$

...8/-

(a) Tuliskan sistem persamaan pembezaan dalam bentuk

$$\frac{dX}{dt} = AX,$$

di mana $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ dan kenalpasti matrik A.

(25 markah)

(b) Tunjukkan nilai eigen bagi A adalah

$$\lambda_1 = 2, \qquad \lambda_2 = i, \qquad \lambda_3 = -i.$$

Seterusnya cari vektor eigen yang bersepadan.

(45 markah)

(c) Cari penyelesaian am bagi sistem berkenaan.

(30 markah)

5. Fungsi $f(t)$ ditakrifkan oleh

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 2t - 3, & t \ge 4 \end{cases}$$

(a) Grafkan fungsi $f(t)$ dari t = 0 ke t = 7.

(10 markah)

(b) Tuliskan fungsi $f(t)$ dalam sebutan fungsi Heaviside $U_a(t)$ di mana $U_a(t)$ ditakrifkan oleh

$$U_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

(10 markah)

(c) Cari jelmaan Laplace bagi $f(t)$.

(30 markah)

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(d) Gunakan Jelmaan Laplace untuk mencari penyelesaian bagi masalah nilai awal

$$y'' + y' = f(t)$$

$$y(0) = 1$$

$$y'(0) = 0$$

di mana $f(t)$ seperti diberi di atas.

[Anda boleh menggunakan pecahan separa berikut

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\frac{1}{s^3(s+1)} = \frac{1}{s^3} - \frac{1}{s^s} + \frac{1}{s} - \frac{1}{s+1}$$

dan keputusan yang diberikan dalam Jadual 1].

(50 markah)

Table 1: Elementary Laplace Transforms

	$f(t)=\mathscr{L}^{-1}\{F(s)\}$	$F(s)=\mathscr{L}\{f(f)\}$
1.	1	$\frac{1}{s},\quad s>0$
2.	e^{at}	$\frac{1}{s-a},\quad s>a$
3.	$t^n,\ n=$ positive integer	$\frac{n!}{s^{n+1}},\quad s>0$
4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}},\quad s>0$
5.	$\sin at$	$\frac{a}{s^2+a^2},\quad s>0$
6.	$\cos at$	$\frac{s}{s^2+a^2},\quad s>0$
7.	$\sinh at$	$\frac{a}{s^2-a^2},\quad s>\lvert a\rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2},\quad s>\lvert a\rvert$
9.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
10.	$e^{ct}f(t)$	$F(s-c)$
11.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

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