UNIVERSITI SAINS MALAYSIA

Final Examination
Academic Session 2008/2009

April 2009

JIM 211 – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains EIGHT printed pages before you begin the examination.

Answer ALL questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.

Jawab SEMUA soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.]

1. (a) Find all the local maxima, local minima and saddle points of the function

$$f(x,y)=x^3+y^3+3x^2-2y^2-8.$$

(40 marks)

(b) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

(30 marks)

(c) Given

$$u^2-v^2+2x+3y=0$$
$$uv+x-y=0.$$

Show that $\dfrac{\partial v}{\partial x}=\dfrac{v-u}{u^2+v^2}$. Hence, find $\dfrac{\partial^2 v}{\partial x^2}$.

(30 marks)

2. (a) Sketch the region of integration for the integral $\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx$ and write an equivalent integral with the order of integration reversed. Evaluate the integral.

(50 marks)

(b) Given the triple integral

$$\iiint\limits_A \left(x^2+y^2\right)^{\frac{3}{2}}dx\,dy\,dz$$
$$A=\left\{(x,y,z):4\le x^2+y^2\le 9,\ 1\le z\le 2\right\}.$$

Sketch the solid A and evaluate the integral.

(50 marks)

3. (a) Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences in $\mathbb{R}$ with

$\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$.

Prove that $\lim_{n\to\infty}(a_n + b_n) = \alpha + \beta$.

(50 marks)

(b) A sequence $\{a_n\}$ is said to be increasing if and only if $a_{n+1} \geq a_n$, for all $n \in \mathbb{N}$. Use the mathematical induction to show that the sequence

$$a_1 = 1,\ a_n = \sqrt{2 + a_{n-1}},\ n > 1$$

is increasing.

(50 marks)

4. (a) State L'Hôpital's Rule for the indeterminate form 0/0 and $\infty.0$. Calculate the following limits.

(i) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$,

(ii) $\lim_{x\to 0^+} \sqrt{x}\,\ell n\, x$,

(iii) $\lim_{x\to\infty} \left(x \sin\dfrac{1}{x} \right)$.

(70 marks)

(b) State the Pinching Theorem for sequences. Hence, use the theorem to evaluate $\lim_{x\to\infty} \dfrac{\cos n\pi}{n}$.

(30 marks)

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5. (a) Determine whether each of the following series converges or diverges.

(i) $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p \geq 2$,

(ii) $\sum_{k=2}^{\infty} \frac{1}{k \, \ell n k}$,

(iii) $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$,

(iv) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$,

(v) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$.

(60 marks)

(b) Find the Taylor's Polynomial of degree n about $x = c$ and the remainder for the function f given by

$$f(x) = x^4 + 2x^3 + 5x^2 - 1, \; n = 3 \text{ and } c = 0.$$

(40 marks)

1. (a) Cari semua titik maksimum tempatan, minimum tempatan dan titik pelana bagi fungsi

$$f(x,y) = x^3 + y^3 + 3x^2 - 2y^2 - 8.$$

(40 markah)

(b) Guna kaedah pendarab Lagrange untuk mencari nilai-nilai maksimum dan minimum bagi fungsi $f(x, y) = 3x + 4y$ pada bulatan $x^2 + y^2 = 1$.

(30 markah)

(c) Diberi

$$u^2 - v^2 + 2x + 3y = 0$$
$$uv + x - y = 0.$$

Tunjukkan bahawa $\dfrac{\partial v}{\partial x} = \dfrac{v-u}{u^2+v^2}$. Dengan ini, cari $\dfrac{\partial^2 v}{\partial x^2}$.

(30 markah)

2. (a) Lakar rantau pengamiran bagi kamiran $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$ dan nyatakan kamiran yang setara setelah penukaran tertib pengamiran dilakukan. Cari nilai kamiran tersebut.

(50 markah)

(b) Diberi kamiran gandatiga

$$\iiint_A \left(x^2 + y^2\right)^{\frac{3}{2}} dx\, dy\, dz$$

$$A = \left\{(x,y,z) : 4 \le x^2 + y^2 \le 9,\ 1 \le z \le 2\right\}.$$

Lakar bongkah A dan dapatkan nilai kamiran tersebut.

(50 markah)

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3. (a) Katakan $\{a_n\}$ dan $\{b_n\}$ adalah dua jujukan dalam $I\!R$ dengan

$\underset{n\to\infty}{\text{had}}\, a_n = \alpha$ dan $\underset{n\to\infty}{\text{had}}\, b_n = \beta.$

Buktikan bahawa $\underset{n\to\infty}{\text{had}}(a_n + b_n) = \alpha + \beta.$

(50 markah)

(b) Jujukan $\{a_n\}$ dikatakan menokok jika dan hanya jika $a_{n+1} \geq a_n$, untuk semua $n \in I\!N$. Guna kaedah aruhan matematik untuk menunjukkan bahawa jujukan

$$a_1 = 1,\ a_n = \sqrt{2 + a_{n-1}},\ n > 1$$

adalah menokok.

(50 markah)

4. (a) Nyatakan Petua L'Hôpital bagi bentuk-bentuk tak tentu 0/0 and ∞.0. Hitung had-had berikut:

(i) $\underset{x\to 0}{\text{had}} \dfrac{\sqrt{1+x} - 1 - \dfrac{x}{2}}{x^2}$,

(ii) $\underset{x\to 0^+}{\text{had}} \sqrt{x}\,\ell n\, x$,

(iii) $\underset{x\to\infty}{\text{had}} \left(x \sin\dfrac{1}{x} \right).$

(70 markah)

(b) Nyatakan Teorem Menyepit bagi jujukan. Dengan ini, guna teorem tersebut untuk mencari nilai $\underset{x\to\infty}{\text{had}} \dfrac{\cos n\pi}{n}$.

(30 markah)

5. (a) Tentukan sama ada setiap siri berikut menumpu atau mencapah.

(i) $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p \geq 2$,

(ii) $\sum_{k=2}^{\infty} \frac{1}{k \ell n k}$,

(iii) $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$,

(iv) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$,

(v) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$.

(60 markah)

(b) Cari Polinomial Taylor berdarjah n pada $x = c$ dan baki bagi fungsi f yang diberikan oleh

$f(x) = x^4 + 2x^3 + 5x^2 - 1$, $n = 3$ and $c = 0$.

(40 markah)

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Lampiran

1. (a) $$\frac{\partial(F_1, F_2, ..., F_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & & \frac{\partial F_n}{\partial x_n} \end{vmatrix}$$

(b) $F(x, y, u, v) = 0$
$G(x, y, u, v) = 0$

$$\frac{\partial v}{\partial x} = -\frac{\partial(F,G)}{\partial(u,x)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

(c) $x = r \cos \theta$
$y = r \sin \theta$
$z = z$

(d) $x = \rho \sin \phi \cos \theta$
$y = \rho \sin \phi \sin \theta$
$z = \rho \cos \phi$

(e) $$M = \iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$\bar{x} = \frac{\iiint_V x\, f(x, y, z)\, dx\, dy\, dz}{M}$$

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